# EUROPEAN BASKETS IN DISCRETE-TIME CONTINUOUS-BINOMIAL MARKET MODELS 

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#### Abstract

We consider a discrete-time incomplete multi-asset market model with continuous price jumps. For a wide class of contingent claims, including European basket call options, we compute the bounds of the interval containing the no-arbitrage prices. We prove that the lower bound coincides, in fact, with Jensen's bound. The upper bound can be computed by restricting to a binomial model for which an explicit expression for the bound is known by an earlier work of the authors. We describe explicitly a maximal hedging strategy which is the best possible in the sense that its value is equal to the upper bound of the price interval of the claim. Our results show that for any $c$ in the interval of the non-arbitrage contingent claim price at time 0 , one can change the boundaries of the price jumps to obtain a model in which $c$ is the upper bound at time 0 of this interval. The lower bound of this interval remains unaffected.


## 1. The main Results

Discrete time continuous-binomial market model. We consider a discrete-time market model, see e.g [2, §4.5] or [8, §3], with $m$ risky assets $S_{1}, \ldots, S_{m}$ and a bond $S_{0}$ whose prices at time $k=0, \ldots, n$ denoted $S_{i}(k)$, are random processes described as follows. We fix the initial values $S_{i}(0)$ of the assets $(i=0,1, \ldots, m)$ and parameters $R>0$ (the interest rate) and $0<D_{i}<R<U_{i}$. For time $k=1,2, \ldots, n$ the random process is defined by

- $S_{0}(k)=R^{k} S_{0}(0)$, and
- $S_{i}(k)=\Psi_{i}(k) S_{i}(k-1)$, for $i=1,2, \ldots, m$, where $\Psi_{i}(k)$ are random variables with values in $\left[D_{i}, U_{i}\right]$.

We call $\Psi_{i}(k)$ the price jumps at time $k$. We emphasize that the price jumps are not assumed to be independent of each other nor identically distributed.

A European basket call option is a contingent claim with pay-off given by

$$
\begin{equation*}
F=\left(\sum_{i=0}^{m} c_{i} \cdot S_{i}(n)-K\right)^{+}, \tag{1}
\end{equation*}
$$

where $K>0$ and $c_{i} \geq 0$ for $i=1,2, \ldots, m$ and $x^{+}:=\max \{x, 0\}$. Notice that there is no assumption on $c_{0}$. The rational values of $F$ at time $k$ are the possible market values of the option at time $k$ so that no arbitrage occurs. They are known to form an open interval $\left(\Gamma_{\min }(F, k), \Gamma_{\max }(F, k)\right)$, where $\Gamma_{\min }(F, k)$ and $\Gamma_{\max }(F, k)$ depend on the "state of the world" at time $k$, namely the history of the market up to time $k$, and in particular they depend on the (current) values of the assets $S_{i}$ at time $k$.

The main result of this paper is the computation of $\Gamma_{\min }(F, k)$ and $\Gamma_{\max }(F, k)$. It extends the main results of the authors' previous work [5 in which we consider discrete time binomial models, i.e ones in which the price jumps $\Psi_{i}(k)$ take values in (the discrete) set $\left\{D_{i}, U_{i}\right\}$ rather than the entire interval $\left[D_{i}, U_{i}\right]$.
1.A. Computation of $\Gamma_{\min }(F, k)$ and $\Gamma_{\max }(F, k)$. For every $1 \leq i \leq m$ set

$$
\begin{equation*}
b_{i}=\frac{R-D_{i}}{U_{i}-D_{i}} \tag{2}
\end{equation*}
$$

and reorder the assets $S_{1}, \ldots, S_{m}$, if necessary, so that

$$
b_{1} \geq \cdots \geq b_{m}
$$

Define $q_{1}, \ldots, q_{m}$ by

$$
q_{i}= \begin{cases}1-b_{1} & \text { if } i=1  \tag{3}\\ b_{i}-b_{i+1} & \text { if } 1<i<m \\ b_{m} & \text { if } i=m\end{cases}
$$

For any $0 \leq i \leq m$ and any $0 \leq j \leq m$ define numbers $\chi_{i}(j)$ as follows.

$$
\begin{align*}
& \chi_{i}(j)=\left\{\begin{array}{ll}
U_{i} & \text { if } i \leq j \\
D_{i} & \text { if } i>j
\end{array} \quad \text { for } 1 \leq i \leq m,\right. \text { and }  \tag{4}\\
& \chi_{0}(j)=R .
\end{align*}
$$

To streamline the notation, set for every $k \geq 0$

$$
\begin{equation*}
\mathcal{P}_{k}(m)=\{0, \ldots, m\}^{k} . \tag{5}
\end{equation*}
$$

Thus, any $J \in \mathcal{P}_{k}(m)$ is a sequence $J=\left(j_{1}, \ldots, j_{k}\right)$ with $0 \leq j_{1}, \ldots, j_{k} \leq m$. For such $J$ set

$$
\begin{align*}
q_{J} & =\prod_{j \in J} q_{j}  \tag{6}\\
\chi_{i}(J) & =\prod_{j \in J} \chi_{i}(j) .
\end{align*}
$$

Theorem 1.1. With the setup of the market model and notation above, the extremal values of the rational values of $F$ at time $0 \leq k \leq n$ are given by

$$
\begin{aligned}
& \Gamma_{\min }(F, k)=R^{k-n} \cdot\left(R^{n-k} \sum_{i=0}^{m} c_{i} \cdot S_{i}(k)-K\right)^{+} \\
& \Gamma_{\max }(F, k)=R^{k-n} \cdot \sum_{J \in \mathcal{P}_{n-k}(m)} q_{J} \cdot\left(\sum_{i=0}^{m} c_{i} \cdot \chi_{i}(J) \cdot S_{i}(k)-K\right)^{+} .
\end{aligned}
$$

1.B. Hedging strategies. Consider a sequence of (time changing) portfolios

$$
V_{\alpha}(k)=\sum_{i} \alpha_{i}(k) S_{i}(k), \quad(0 \leq k \leq n-1)
$$

for some choices of values for $\alpha_{i}(k)$ at time $0 \leq k \leq n-1$. A maximum hedging strategy is a choice for $\alpha_{i}(k)$ at time $0 \leq k \leq n-1$ (which depends on the state of the world at that time) which minimizes the value $V_{\alpha}(k)$ subject to the requirement that

$$
\sum_{i=0}^{m} \alpha_{i}(k) \cdot S_{i}(k+1) \geq \Gamma_{\max }(F, k+1)
$$

for any subsequent state of the world at time $k+1$. That is, a maximum hedging strategy is a time dependant portfolio of minimum possible cost whose value is guaranteed to exceed the future rational value of the contingent claim $F$.

Our next result, Theorem 1.2, shows that the value of any hedging portfolio $V_{\beta}(k)$ must always exceed $\Gamma_{\max }(F, k)$ and that there exists a hedging strategy $\alpha_{i}(k)$ that attains this
bound. It is a minimum cost maximum hedging strategy. To make this result precise, given the state of the world at time $0 \leq k \leq n-1$, let $Y_{i}(k)$ where $0 \leq i \leq m$ be the value of $\Gamma_{\max }(F, k+1)$ at the state of the world at time $k+1$ which is obtained from the present one (at time $k$ ) by having the assets $S_{1}, \ldots, S_{i}$ make their maximum price jumps $U_{1}, \ldots, U_{i}$ and having $S_{i+1}, \ldots, S_{m}$ make their minimum price jumps $D_{i+1}, \ldots, D_{m}$. Explicitly, for any $0 \leq t \leq m$ :

$$
Y_{t}(k)=R^{k+1-n} \cdot \sum_{J \in \mathcal{P}_{n-k-1}(m)} q_{J}\left(\sum_{i=0}^{m} c_{i} \cdot \chi_{i}(J) \chi_{i}(t) \cdot S_{i}(k)-K\right)^{+} .
$$

Theorem 1.2. Consider the continuous binomial market model above and a European option $F$. Any hedging strategy $\beta_{i}(k)$ satisfies

$$
V_{\beta}(k) \geq \Gamma_{\max }(F, k)
$$

There exists a maximum hedging strategy $\alpha_{i}(0), \ldots, \alpha_{i}(n-1)$ such that $V_{\alpha}(k)=\Gamma_{\max }(F, k)$ for all $0 \leq k \leq n-1$. In fact, the values of $\alpha_{i}(k)$ at time $k$ are computed as follows.

$$
\left[\begin{array}{c}
\alpha_{0}(k) \\
\alpha_{1}(k) \\
\vdots \\
\alpha_{m}(k)
\end{array}\right]=W(k) \cdot N \cdot Q \cdot\left[\begin{array}{c}
Y_{0}(k) \\
Y_{1}(k) \\
\vdots \\
Y_{m}(k)
\end{array}\right]
$$

Where $W(k), N, T$ are the following $(m+1) \times(m+1)$ matrices. Set $\Delta_{i}=U_{i}-D_{i}$.

$$
\begin{aligned}
& W(k)=\left[\begin{array}{cccc}
\frac{1}{R \cdot S_{0}(k)} & & & \\
& \frac{1}{S_{1}(k)} & & \\
& & \ddots & \\
& & & \frac{1}{S_{m}(k)}
\end{array}\right] \\
& Q=\left[\begin{array}{ccclcc}
1 & 0 & 0 & \cdots \cdots & 0 & 0 \\
-1 & 1 & 0 & \cdots \cdots & 0 & 0 \\
0 & -1 & 1 & \cdots \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots \cdots & & \vdots \\
0 & 0 & 0 & \cdots \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots \cdots & -1 & 1
\end{array}\right] \\
& N=\left[\begin{array}{c|cccc}
1 & -\frac{D_{1}}{\Delta_{1}} & -\frac{D_{2}}{\Delta_{2}} & \cdots \cdots & -\frac{D_{m}}{\Delta_{m}} \\
\hline 0 & \frac{1}{\Delta_{1}} & 0 & \cdots \cdots & 0 \\
0 & 0 & \frac{1}{\Delta_{2}} & \cdots \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & \cdots \cdots & \frac{1}{\Delta_{m}}
\end{array}\right]
\end{aligned}
$$

(Notice that $W(k)$ depends on the state of the world, but $N$ and $Q$ do not. Also, $R \cdot S_{0}(k)=$ $\left.S_{0}(0) \cdot R^{k+1}\right)$.

This result extends our previous result in [6] which computes a similar hedging strategy in discrete time binomial models, i.e models in which $\Psi_{i}(k) \in\left\{D_{i}, U_{i}\right\} \subset\left[D_{i}, U_{i}\right]$.
Changing the parameters of the model. Keeping $R$ fixed, we may change the values of $U_{i}$ and $D_{i}$ to obtain different models for the same market. This has the effect of changing the limits of the price jumps of the assets $S_{i}$, and consequently the random processes $S_{i}$ are changed. Clearly the values of $\Gamma_{\min }(F, k)$ and $\Gamma_{\max }(F, k)$ depend on these parameters, and
we therefore write $\Gamma_{\min }\left(F, k ; U_{i}, D_{i}\right)$ and $\Gamma_{\max }\left(F, k ; U_{i}, D_{i}\right)$ to emphasise this dependence. We will be interested in the rational prices of $F$ at time 0 , namely $\Gamma_{\min }\left(F, 0 ; U_{i}, D_{i}\right)$ and $\Gamma_{\text {max }}\left(F, 0 ; U_{i}, D_{i}\right)$.

Theorem 1.3. Consider a market model with some $0<D_{i}<R<U_{i}$ and the European basket $F$ in (1). Then
(1) $\Gamma_{\min }\left(F, 0 ; U_{i}, D_{i}\right)=\left(R^{n} \sum_{i=0}^{m} c_{i} S_{i}(0)-K\right)^{+}$and in particular it is independent of the values of $U_{i}, D_{i}$.
(2) For every $c$ in the open interval $\left(\Gamma_{\min }\left(F, 0 ; U_{i}, D_{i}\right), \Gamma_{\max }\left(F, 0 ; U_{i}, D_{i}\right)\right)$ there exist $D_{i} \leq d_{i}<R<u_{i} \leq U_{i}$ such that $\Gamma_{\max }\left(F, 0 ; u_{i}, d_{i}\right)=c$.

More precisely, consider the functions $u_{i}, d_{i}:[0,1) \rightarrow \mathbb{R}$ defined by

$$
\begin{aligned}
d_{i}(s) & =D_{i}+\left(R-D_{i}\right) s \\
u_{i}(s) & =\frac{R-\left(1-b_{i}\right) d_{i}(s)}{b_{i}}
\end{aligned}
$$

Then $\varphi(s)=\Gamma_{\max }\left(F, 0 ; u_{i}(s), d_{i}(s)\right)$ is a continuous function of $s \in[0,1)$ such that $\varphi(0)=\Gamma_{\max }\left(F, 0 ; U_{i}, D_{i}\right)$ and $\lim _{s \nearrow 1} \varphi(s)=\Gamma_{\min }\left(F, 0 ; U_{i}, D_{i}\right)$.

## 2. Preliminaries: Random processes and conditional expectation

2.A. Non-degenerate density functions. This section is concerned with some general results about probability measure spaces. Our standard reference for Measure theory and Lebesgue integration are Halmos [4] and Royden [9], and for Probability Theory it is Feller [1].

Throughout this paper, once $m \geq 1$ is fixed we will denote

$$
\begin{equation*}
\Omega=[0,1]^{m} \subseteq \mathbb{R}^{m} \tag{7}
\end{equation*}
$$

equipped with the usual Borel $\sigma$-algebra and probability measure $\mu$.
A probability density function ( $p d f$ ) is a measurable $p: \Omega \rightarrow[0, \infty)$ such that $\int_{\Omega} p d \mu=1$. It gives rise in a standard way to a probability measure on $\Omega$ which by abuse of notation we also denote by $p$.

A probability measure on $\Omega$ is called absolutely continuous with respect to $\mu$, written $\nu \ll \mu$, if for any Borel subset $E$ we have $\mu(E)=0 \Longrightarrow \nu(E)=0$. A probability measure $\nu$ on $\Omega$ is non-degenerate if $\nu \ll \mu$ and $\mu \ll \nu$. We write $\nu \approx \mu$.

By the Radon-Nykodim theorem [9, §11.5] if $\nu \ll \mu$ then there exists a pdf $p: \Omega \rightarrow[0, \infty)$ called the Radon-Nykodim derivative, such that $\nu(E)=\int_{E} p(x) d \mu(x)$. It is easy to check that $\nu \approx \mu$ if and only if $p>0$ almost everywherd ${ }^{11}$. We say that $p$ is non-degenerate.
2.B. Conditional probability. Consider $\Omega_{(i)}=[0,1]^{m_{i}}$ where $i=1, \ldots, n$. Set

$$
\Omega=\prod_{i=1}^{n} \Omega_{(i)}=[0,1]^{m}
$$

where $m=\sum_{i} m_{i}$, equipped with the standard Borel $\sigma$-algebra. Let $X_{(i)}$ denote the random vector

$$
X_{(i)}: \Omega \xrightarrow{\operatorname{proj}_{i}} \Omega_{(i)} \subseteq \mathbb{R}^{m_{i}} .
$$

[^0]For a non-empty $I \subseteq\{1, \ldots, n\}$ set $m_{I}=\sum_{i \in I} m_{i}$. Then set

$$
\begin{aligned}
& \Omega_{(I)}=\prod_{i \in I} \Omega_{(i)} \\
& X_{(I)}: \Omega \xrightarrow{\operatorname{proj}_{(I)}} \Omega_{(I)} \subseteq \mathbb{R}^{m_{I}} .
\end{aligned}
$$

Thus, $\Omega_{(I)}=[0,1]^{m_{I}}$ and $X_{(I)}$ is a random vector into $\mathbb{R}^{m_{I}}$. We will denote elements of $\Omega_{(I)}$ by $\omega_{(I)}$. We will write $n-I$ for the complement of $I$.

For the remainder of this subsection we fix a non-degenerate (with respect to the Lebesgue measure on $\Omega$ ) pdf $p: \Omega \rightarrow[0, \infty)$ and equip $\Omega$ with the probability measure it induces which we abusively denote by $p$. Note that $p$ is the joint density function of the random vectors $X_{(1)}, \ldots, X_{(n)}$ (because $X_{\{1, \ldots, n\}}$ is the inclusion $\Omega \subseteq \mathbb{R}^{m}$ ).

Given a non-empty $I \subseteq\{1, \ldots, n\}$, the (joint) density function of $X_{(I)}$ is the function $p_{X_{(I)}}: \Omega_{(I)} \rightarrow[0, \infty)$ given by (See e.g. 33, Chap. 2, Scet. 3])

$$
\begin{equation*}
p_{X_{(I)}}\left(\omega_{(I)}\right)=\int_{\tau \in \Omega_{(n-I)}} p\left(\omega_{(I)}, \tau\right) d \tau \tag{8}
\end{equation*}
$$

Fubini's theorem readily implies that $p_{X_{(I)}}$ is non-degenerate (with respect to the Lebesgue measure on $\Omega_{(I)}$ ).

Consider some disjoint $I, J \subseteq\{1, \ldots, n\}$. The density function of $X_{(I)}$ given $X_{(J)}$ denoted $p_{X_{(I)} \mid X_{(J)}}: \Omega_{(I \cup J)} \rightarrow[0, \infty)$ is

$$
\begin{equation*}
p_{X_{(I)} \mid X_{(J)}}\left(\omega_{(I)}, \omega_{(J)}\right)=\frac{p_{X_{(I U J)}}\left(\omega_{(I)}, \omega_{(J)}\right)}{p_{X_{(J)}\left(\omega_{(J)}\right)}} \tag{9}
\end{equation*}
$$

whenever this is defined. Since $p_{X_{(J)}}$ is non-degenerate, $p_{X_{(I)} \mid X_{(J)}}$ is defined a.e. For any $\omega_{(J)}$ we obtain a function, the (conditional) density of $X_{(I)}$ given the event $\left\{X_{(J)}=\omega_{(J)}\right\}$,

$$
p_{X_{(I)} \mid X_{(J)}=\omega_{(J)}}: \Omega_{(I)} \rightarrow[0, \infty)
$$

defined by $p_{X_{(I)} \mid X_{(J)}=\omega_{(J)}}(-)=p_{X_{(I)} \mid X_{(J)}}\left(-, \omega_{(J)}\right)$. By Fubini's theorem it is a (measurable) probability density function on $\Omega_{(I)}$.

Consider a random vector

$$
f: \Omega \rightarrow \mathbb{R}^{k}
$$

To avoid issues of convergence we assume that $f \geq 0$, namely all the components of $f$ are non-negative. The expectation of $f$ given $X_{(I)}$ is the function

$$
E_{p}\left(f \mid X_{(I)}\right): \Omega_{(I)} \rightarrow \mathbb{R}
$$

where $E_{p}\left(f \mid X_{(I)}\right)\left(\omega_{(I)}\right)$ is the conditional expectation $E_{p}\left(f \mid X_{(I)}=\omega_{(I)}\right)$, namely

$$
E_{p}\left(f \mid X_{(I)}\right)\left(\omega_{(I)}\right)=\int_{\tau \in \Omega_{(n-I)}} f\left(\tau, \omega_{(I)}\right) p_{X_{(n-I)} \mid X_{(I)}}\left(\tau, \omega_{(I)}\right) d \tau
$$

By Fubini's theorem $E_{p}\left(f \mid X_{(I)}\right)$ is a measurable function.
Lemma 2.1. Keeping the notation above, let $I, J \subseteq\{1, \ldots, n\}$ be disjoint and consider some $\omega_{(I)} \in \Omega_{(I)}$. Set $p^{\prime}=p_{X_{(J)} \mid X_{(I)}=\omega_{(I)}}$, a pdf on $\Omega_{(J)}$. Let $f \geq 0$ be a random vector on $\Omega$. Then

$$
E_{p}\left(f \mid X_{(I)}=\omega_{(I)}\right)=E_{p^{\prime}}\left(\tau \mapsto E_{p}\left(f \mid X_{(I \cup J)}\right)\left(\omega_{(I)}, \tau\right)\right) .
$$

In particular,

$$
E_{p}\left(X_{(J)} \mid X_{(I)}=\omega_{(I)}\right)=E_{p^{\prime}}\left(X_{(J)}\right)
$$

where $X_{(J)}$ is viewed as a random vector from $\Omega_{(J)}$.

Proof. We compute the right hand side using Fubini's theorem as follows

$$
\begin{aligned}
E_{p^{\prime}}(\tau \mapsto & \left.E_{p}\left(f \mid X_{(I \cup J)}\right)\left(\omega_{(I)}, \tau\right)\right)= \\
& =\int_{\tau \in \Omega_{(J)}} p_{X_{(J)} \mid X_{(I)}=\omega_{(I)}}(\tau) \cdot E_{p}\left(f \mid X_{(I \cup J)}\right)\left(\omega_{(I)}, \tau\right) d \tau \\
& =\int_{\tau \in \Omega_{(J)}}\left(\frac{p_{X_{(I \cup J)}\left(\omega_{(I)}, \tau\right)}}{p_{X_{(I)}}\left(\omega_{(I)}\right)} \cdot \int_{\theta \in \Omega_{(n-I \cup J)}} f\left(\omega_{(I)}, \tau, \theta\right) \cdot \frac{p\left(\omega_{(I)}, \tau, \theta\right)}{\left.p_{\left.X_{(I \cup J)}\right)} \omega_{(I)}, \tau\right)} d \theta\right) d \tau \\
& =\int_{\tau \in \Omega_{(J)}} \int_{\theta \in \Omega_{(n-I \cup J)}} f\left(\omega_{(I)}, \tau, \theta\right) \cdot \frac{p\left(\omega_{(I)}, \tau, \theta\right)}{p_{X_{(I)}\left(\omega_{(I)}\right)}} d \theta d \tau \\
& =\int_{\omega \in \Omega_{(n-I)}} f\left(\omega_{(I)}, \omega\right) \cdot \frac{p\left(\omega_{(I)}, \omega\right)}{p_{X_{(I)}} \omega_{(I))}} d \omega \\
& =\int_{\omega \in \Omega_{(n-I)}} f\left(\omega_{(I)}, \omega\right) \cdot p_{X_{(n-I)} \mid X_{(I)}=\omega_{(I)}}(\omega) d \omega \\
= & E_{p}\left(f \mid X_{(I)}=\omega_{(I)}\right)
\end{aligned}
$$

This establishes the first claim. We apply it to the random vector $f=X_{(J)}$ to obtain the second claim as follows

$$
E_{p}\left(X_{(J)} \mid X_{(I)}=\omega_{(I)}\right)=E_{p^{\prime}}\left(\tau \mapsto E_{p}\left(X_{(J)} \mid X_{(I \cup J)}\right)\left(\omega_{(I)}, \tau\right)\right)=E_{p^{\prime}}\left(\tau \mapsto X_{(J)}(\tau)\right)=E_{p^{\prime}}\left(X_{(J)}\right)
$$

Here we observe that $E_{p}\left(X_{(J)} \mid X_{(I \cup J)}\right)\left(\omega_{(I)}, \tau\right)=X_{(J)}(\tau)$ because with the abuse of notation for the domain of $X_{(J)}$ we have $X_{(J)}\left(\omega_{(I)}, \tau, \theta\right)=X_{(J)}(\tau)$ for any $\tau \in \Omega_{(J)}$ and any $\theta \in$ $\Omega_{(n-I \cup J)}$.

Lemma 2.2. Let $f: \Omega \rightarrow \mathbb{R}^{d}$ be a function and $I, J \subseteq\{1, \ldots, n\}$ be disjoint, and assume that $f \geq 0$. Suppose that factors through the projection $\Omega \xrightarrow{\pi_{I \cup J}} \Omega_{(I \cup J)}$, namely there exists $g: \Omega_{(I \cup J)} \rightarrow \mathbb{R}^{m}$ such that $f=g \circ \pi_{I \cup J}$. Set $p^{\prime}=p_{X_{(J)} \mid X_{(I)}=\omega_{(I)}}$, pdf on $\Omega_{(J)}$. Then

$$
E_{p}\left(f \mid X_{(I)}=\omega_{(I)}\right)=E_{p^{\prime}}\left(\omega_{(J)} \mapsto g\left(\omega_{(I)} \omega_{(J)}\right)\right)
$$

Proof. Lemma 2.1 gives

$$
E_{p}\left(f \mid X_{(I)}=\omega_{(I)}\right)=E_{p^{\prime}}\left(\omega_{(J)} \mapsto E_{p}\left(f \mid X_{(I \cup J)}\right)\left(\omega_{(I)}, \omega_{(J)}\right)\right)=E_{p^{\prime}}\left(\omega^{J} \mapsto g\left(\omega^{I}, \omega^{J}\right)\right)
$$

because $f\left(\omega_{(I)}, \omega_{(J)}, \tau\right)=g\left(\omega_{(I)}, \omega_{(J)}\right)$ for any $\tau \in \Omega_{(n-I \cup J)}$, so $E_{p}\left(f \mid X_{(I \cup J)}\right)\left(\omega_{(I)}, \omega_{(J)}\right)=$ $g\left(\omega_{(I)}, \omega_{(J)}\right)$.
2.C. Tensoring. Given $p: \Omega \rightarrow \mathbb{R}$ and $q: \Omega^{\prime} \rightarrow \mathbb{R}$ we obtain a function $p \otimes q: \Omega \times \Omega^{\prime} \rightarrow \mathbb{R}$ by

$$
\begin{equation*}
(p \otimes q)\left(\omega, \omega^{\prime}\right)=p(\omega) \cdot q\left(\omega^{\prime}\right) \tag{10}
\end{equation*}
$$

It is clear that if $p, q$ are pdf's then so is $p \otimes q$ and that it is non-degenerate if $p$ and $q$ are non-degenerate.

Keeping the notation above for $\Omega_{(i)}$ and the random vectors $X_{(i)}$, let $p_{(i)}: \Omega_{(i)} \rightarrow[0, \infty)$ be non-degenerate pdf's. Then $p=p_{(1)} \otimes \cdots \otimes p_{(n)}$ is a non-degenerate pdf on $\Omega=\prod_{i} \Omega_{(i)}$. For any $I \subseteq\{1, \ldots, n\}$ we denote

$$
p_{(I)}=\otimes_{i \in I} p_{(i)}
$$

This is a non-degenerate pdf on $\Omega_{(I)}$.
Lemma 2.3. Let $p_{(i)}: \Omega_{(i)} \rightarrow[0, \infty)$ be non-degenerate $p d f$ 's, $i=1, \ldots, n$. Set $p=p_{(1)} \otimes$ $\cdots \otimes p_{(n)}$, non-degenerate pdf on $\Omega$. Then
(1) $p_{X_{(I)}}=p_{(I)}$ for any $I \subseteq\{1, \ldots, n\}$.
(2) $p_{X_{(J)} \mid X_{(I)}=\omega_{(I)}}=p_{(J)}$ for any disjoint $I, J \subseteq\{1, \ldots, n\}$, and furthermore
(3) $E_{p}\left(X_{(J)} \mid X_{(I)}=\omega_{(I)}\right)=E_{p_{(J)}}\left(X_{(J)}\right)$ where $X_{(J)}$ is viewed as a random vector on $\Omega_{(J)}$.

Proof. (11) Given $\omega_{(I)}$ we compute

$$
p_{X_{(I)}}\left(\omega_{(I)}\right)=\int_{\tau \in \Omega_{(n-I)}} p\left(\omega_{(I)}, \tau\right) d \tau=\int_{\tau \in \Omega_{(n-I)}} p_{(I)}\left(\omega_{(I)}\right) \cdot p_{(n-I)}(\tau) d \tau=p_{(I)}\left(\omega_{(I)}\right)
$$

because $p_{(J)}$ is a pdf on $\Omega_{(J)}$ for any $J$.
(2) Given $\omega_{(J)}$ and $\omega_{(I)}$ we use this to compute

$$
\begin{aligned}
& p_{X_{(J)} \mid X_{(I)}=\omega^{I}}\left(\omega_{(J)}\right)=\frac{p_{X_{(I \cup J)}}\left(\omega_{(I)}, \omega_{(J)}\right)}{p_{X_{(I)}}\left(\omega_{(I)}\right)}=\frac{p_{(I \cup J)}\left(\omega_{(I)}, \omega_{(J)}\right)}{p_{(I)}\left(\omega_{(I)}\right)} \\
&=\frac{p_{(I)}\left(\omega_{(I)}\right) \cdot p_{(J)}\left(\omega_{(J)}\right)}{p_{(I)}\left(\omega_{(I)}\right)}=p_{(J)}\left(\omega_{(J)}\right) .
\end{aligned}
$$

(3) By Lemma 2.1 and item (2)

$$
\begin{aligned}
E_{p}\left(X_{(J)} \mid X_{(I)}=\omega_{(I)}\right)=E_{p_{X_{(J)} \mid X_{(I)}=\omega_{(I)}}\left(\omega_{(J)} \mapsto E_{p}\left(X_{(J)} \mid X_{(I \cup J)}\left(\omega_{(I)}, \omega_{(J)}\right)\right)\right.} \\
=E_{p_{(J)}}\left(\omega_{(J)} \mapsto X_{(J)}\left(\omega_{(J)}\right)\right) .
\end{aligned}
$$

## 2.D. Processes.

Definition 2.4. Let $\Omega$ be a set and $n \geq 1$. An $\mathbb{R}^{d}$-valued process (over $\Omega$ ) is a sequence of functions

$$
Y^{(0)}, \ldots, Y^{(n)}: \Omega^{n} \rightarrow \mathbb{R}^{d}
$$

such that for each $0 \leq k \leq n$ the function $X^{(k)}$ factors through the projection $\Omega^{n} \rightarrow \Omega^{k}$ to the first $k$ factors.

If $\Omega^{n}$ is equipped with a probability measure, $Y^{(0)}, \ldots, Y^{(k)}$ is called an $\mathbb{R}^{d}$-valued random process.

We frequently regard $Y^{(k)}$ as a function with domain $\Omega^{k}$. We will sometimes "trim" the process to $Y^{(1)}, \ldots, Y^{(n)}$ or to $X^{(0)}, \ldots, X^{(n-1)}$.

If $\Omega=[0,1]^{m} \subseteq \mathbb{R}^{m}$ the projections $L^{(k)}: \Omega^{n} \rightarrow \Omega \subseteq \mathbb{R}^{m}$ to the $k$-th factor give a universal process in the sense that if $Y^{(0)}, \ldots, Y^{(n)}$ is a process then each $Y^{(k)}$ is a function of $L^{(1)}, \ldots, L^{(k)}$.
2.E. Supports. Let $\nu$ be a probability measure on a set $\Omega$.

Definition 2.5. We say that $\nu$ is supported on a measurable set $A$ if $\mu(A)=1$.
Any probability measure $\nu$ on $A \subseteq \Omega$ extends to a probability measure $\tilde{\nu}$ on $\Omega$ supported on $A$ by $\tilde{\nu}(E)=\nu(A \cap E)$. Conversely, if $\nu$ on $\Omega$ is supported by $A$ then $\nu=\widetilde{\left.\nu\right|_{A}}$. If $A$ is finite then for any $f: \Omega \rightarrow \mathbb{R}$ we have $E_{\nu}(f)=\sum_{a \in A} \nu(\{a\}) f(a)$.

## 3. MAXimum and minimum expectation of Random variables on $\Omega=[0,1]^{m}$

Fix some $m \geq 1$ and $\Omega=[0,1]^{m}$ equipped with the usual Lebesgue measure $\mu$. Let

$$
L: \Omega \rightarrow \mathbb{R}^{m}
$$

denote the inclusion. We think of it as a random vector with components $L=\left(\ell_{1}, \ldots, \ell_{m}\right)$. Thus, $\ell_{i}: \Omega \rightarrow \mathbb{R}$ is the projection to the $i$ th factor

$$
\ell_{i}\left(x_{1}, \ldots, x_{m}\right)=x_{i}
$$

For convenience we also set

$$
\ell_{0}=1
$$

the constant function (random variable).
Definition 3.1. Let $\mathcal{M}(\Omega)$ denote the set of all probability measures on $\Omega$ on the Borel $\sigma$-algebra. Let $\mathcal{M}^{+}(\Omega)$ denote the set of all the non-degenerate probability measures (with respect to the Lebesgue measure).

We will often refer to the elements of $\mathcal{M}^{+}(\Omega)$ as pdf's which are non-vanishing a.e.
Consider a non decreasing $b \in \operatorname{int}(\Omega)=(0,1)^{m}$, namely

$$
1>b_{1} \geq \cdots \geq b_{m}>0
$$

We will also denote for convenience

$$
b_{0}=1 \quad \text { and } \quad b_{m+1}=0 .
$$

Definition 3.2. The set of mean-b probability measures on $\Omega$ is

$$
\mathcal{M}(\Omega, b)=\left\{P \in \mathcal{M}(\Omega): E_{P}(L)=b\right\} .
$$

That is, $E_{P}\left(\ell_{i}\right)=b_{i}$ for all $1 \leq i \leq m$. The set of non-degenerate mean- $b$ probability measures is

$$
\mathcal{M}^{+}(\Omega, b)=\left\{P \in \mathcal{M}^{+}(\Omega): E_{P}(L)=b\right\} .
$$

Definition 3.3. Let $\mathcal{L}$ denote the set of vertices of the cube $\Omega=[0,1]^{m}$. That is,

$$
\mathcal{L}=\{0,1\}^{m} .
$$

There is a standard identification of $\mathcal{L}$ with $\wp(\{1, \ldots, m\})$ where $\lambda \in \mathcal{L}$ corresponds to $\operatorname{supp}(\lambda)$. This turns $\mathcal{L}$ into a lattice where the partial order $\preceq$ is induced by inclusion of sets and joins and meets are $\cup$ and $\cap$. The next concept is originally due to Lovász [7].

Definition 3.4. A function $f: \mathcal{L} \rightarrow \mathbb{R}$ is called supermodular if for any $a, b \in \mathcal{L}$

$$
f(a \vee b)+f(a \wedge b) \geq f(a)+f(b)
$$

It is called modular if equality holds.
Definition 3.5. A function $f: \Omega \rightarrow \mathbb{R}$ is called convex-supermodular if $f$ is convex and its restriction to $\mathcal{L}$ is supermodular.

Example 3.6. Let $f: \Omega \rightarrow \mathbb{R}$ and suppose that $f=h \circ g$ for some $g:[0,1]^{m} \rightarrow \mathbb{R}$ and $h: \mathbb{R} \rightarrow \mathbb{R}$ such that either
(1) $h$ is convex and $g$ is affine with non-negative coefficients except the constant term, namely $g=\sum_{i=0}^{m} a_{i} \ell_{i}$ where $a_{1}, \ldots, a_{m} \geq 0$.
(2) $g$ is convex, $\left.g\right|_{\mathcal{L}}$ is modular, and $h$ is convex and increasing.

Then $f$ is convex-supermodular.
Proof: Indeed, $f$ is convex as composition of convex functions and $\left.f\right|_{\mathcal{L}}$ is supermodular by [Simchi-Levi, Theorem 2.2.6] for item (11) and [Simchi-Levi Proposition 2.2.5(c)] for item (2). $\diamond$

For every $0 \leq k \leq m$ let $\rho_{k} \in \mathcal{L}$ denote the element corresponding to $\{1, \ldots, k\}$, namely

$$
\begin{equation*}
\rho_{k}=(\underbrace{1, \ldots, 1}_{k \text { times }}, 0, \ldots, 0) \in\{0,1\}^{m} . \tag{11}
\end{equation*}
$$

Definition 3.7 (Compare [5]). The upper supermodular vertex is the probability density function $q^{*}: \mathcal{L} \rightarrow \mathbb{R}$ supported on $\left\{\rho_{0}, \ldots, \rho_{m}\right\}$ with

$$
q^{*}\left(\rho_{k}\right)=b_{k}-b_{k+1}, \quad(0 \leq k \leq m) .
$$

One easily checks that $\sum_{i=0}^{m} q^{*}\left(\rho_{k}\right)=1$ and that $E_{q^{*}}\left(\ell_{i}\right)=b_{i}$, thus
Proposition 3.8. $q^{*} \in \mathcal{M}(\Omega, b)$ and it is supported on $\mathcal{L} \subseteq \Omega$.
The main result of this section is the following theorem.
Theorem 3.9. Let $f: \Omega \rightarrow \mathbb{R}$ be a continuous convex-supermodular function and assume that $f \geq 0$. Let $q^{*}$ be the upper supermodular vertex of $\mathcal{L}$ (Definition (3.7)). Then

$$
\sup _{p \in \mathcal{M}^{+}(\Omega, b)} E_{p}(f)=E_{q^{*}}\left(\left.f\right|_{\mathcal{L}}\right)=\max _{p \in \mathcal{M}(\Omega, b)} E_{p}(f) .
$$

Note that $E_{q^{*}}\left(\left.f\right|_{\mathcal{L}}\right)=\sum_{i=0}^{m} q^{*}\left(\rho_{i}\right) \cdot f\left(\rho_{i}\right)$.
In the remainder of this section we prove this theorem. It relies on the following key observation. Equip $\mathbb{R}^{m}$ with the norm $\left\|\|_{\infty}\right.$ and restrict this norm to $\Omega=[0,1]^{m}$.

Lemma 3.10 (Approximation lemma). Consider some $q \in \mathcal{M}(\Omega, b)$ supported on a finite subset $\left\{x^{1}, \ldots, x^{k}\right\}$ of $\Omega$ and set $q_{i}=q\left(\left\{x^{i}\right\}\right)$. Suppose that $b \in \operatorname{int}(\Omega)=(0,1)^{m}$. Then for any $\epsilon>0$ and $\delta>0$ there exists $p \in \mathcal{M}^{+}(\Omega, b)$ and $\beta<\epsilon$ such that for any continuous $f: \Omega \rightarrow \mathbb{R}$

$$
E_{p}(f)=\beta \int_{\Omega} f d \mu+\sum_{i=0}^{k}\left(q_{i}-\frac{\beta}{k}\right) \cdot f\left(\xi^{i}\right)
$$

where $\xi^{1}, \ldots, \xi^{k} \in \Omega$ are such that $\left\|\xi^{i}-x^{i}\right\|_{\infty}<\delta$.
Proof. Closed balls of radius $r>0$ in $\mathbb{R}^{m}$ have the form $B(y, r)=y+[-r, r]^{m}$ so their volume, hence their Lebesgue measure, is $(2 r)^{m}$. If $y=\left(y_{1}, \ldots, y_{m}\right)$ then by inspection, for any $1 \leq j \leq m$

$$
\int_{B(y, r)} x_{j} d \mu\left(x^{1}, \ldots, x^{m}\right)=(2 r)^{m} y_{j} .
$$

Claim: There exist distinct $y^{1}, \ldots, y^{k}$ in $\operatorname{int}(\Omega)=(0,1)^{m}$ such that $\left\|y^{i}-x^{i}\right\|_{\infty}<\frac{\delta}{3}$ and such that $\sum_{i=1}^{k} q_{i} y^{i}=b$.
Proof: We show how to perturb the vectors $x^{1}, \ldots, x^{k}$ in order to obtain $y^{1}, \ldots, y^{k}$. First, $\sum_{i=1}^{k} q_{i} x^{i}=E_{q}(L)=b$ since $q \in \mathcal{M}(\Omega, b)$. Suppose that for some $1 \leq j \leq m$ not all of $x_{j}^{1}, \ldots, x_{j}^{k}$ are in the open interval $(0,1)$. If $x_{j}^{i^{\prime}}=0$ for some $i^{\prime}$ then there must exist $i^{\prime \prime}$ such that $x_{j}^{i^{\prime \prime}}>0$ because $\sum_{i=1}^{k} q_{i} x_{j}^{i}=b_{j}>0$. Since $q_{i}>0$ for all $i$ we can increase $x_{j}^{i^{\prime}}$ and decrease $x_{j}^{i^{\prime \prime}}$ by a small positive number $<\frac{\delta}{3}$ so that the sum remains $b_{j}$ and that the new
values of $x_{j}^{i^{\prime}}$ and $x_{j}^{i^{\prime \prime}}$ are in $(0,1)$. Similarly, if $x_{j}^{i^{\prime}}=1$ for some $i^{\prime}$ then there must exist some $i^{\prime \prime}$ such that $x_{j}^{i^{\prime \prime}}<1$ because $\sum_{i=1}^{k} q_{i} x_{j}^{i}=b_{j}<1$. We can then decrease $x_{j}^{i^{\prime}}$ and increase $x_{j}^{i^{\prime \prime}}$ by at most $\frac{\delta}{3}$ so that the sum remains $b_{j}$ and the new values of $x_{j}^{i^{\prime}}$ and $x_{j}^{i_{j}^{\prime \prime}}$ are in $(0,1)$. By repeating this process we can perturb $x^{1}, \ldots, x^{k}$ into $y^{1}, \ldots, y^{k}$ in int $(\Omega)$ such that $\sum_{i} q_{i} y^{i}=b$ and $\left\|x^{i}-y^{i}\right\|_{\infty}<\frac{\delta}{3}$. Since the $x^{i}$ s admit pairwise disjoint neighbourhoods, we can perturb the $x^{i}$ 's inside these neighbourhoods to make sure that the $y^{i}$ 's are distinct. q.e.d

Since $y^{1}, \ldots, y^{k}$ are in the interior of $\Omega$ there exists $r<\frac{\delta}{3}$ such that $B\left(y^{i}, 2 r\right) \subseteq(0,1)^{m}$ for all $1 \leq i \leq k$. Thus, if $\gamma \in \mathbb{R}^{m}$ is such that $\|\gamma\|_{\infty}<r$ then $B\left(y^{i}+\gamma, r\right) \subseteq(0,1)^{m}$. Set

$$
Q=\min \left\{q_{1}, \ldots, q_{m}\right\} .
$$

For any $\beta>0$ set

$$
\gamma_{j}^{1}:=\frac{\beta \cdot\left(\frac{1}{k} \sum_{i=1}^{k} y_{j}^{i}-\frac{1}{2}\right)}{q_{1}-\frac{\beta}{k}} .
$$

Choose $0<\beta<\min \{\epsilon, k Q\}$ sufficiently small such that for every $1 \leq j \leq m$

$$
\left|\gamma_{j}^{1}\right|<r .
$$

Let $\gamma^{1} \in \mathbb{R}^{m}$ be the vector with the components $\gamma_{j}^{1}$ defined above and let $\gamma^{2}, \ldots, \gamma^{k} \in \mathbb{R}^{m}$ be the zero vectors. By construction $\left\|\gamma^{i}\right\|_{\infty}<r$ for all $1 \leq i \leq k$, hence $B\left(y^{i}+\gamma^{i}, r\right) \subseteq(0,1)^{m}$. Define $p: \Omega \rightarrow \mathbb{R}$ by

$$
p=\beta+\sum_{i=1}^{k} \frac{q_{i}-\frac{\beta}{k}}{(2 r)^{m}} \cdot \mathbf{1}_{B\left(y^{i}+\gamma^{i}, r\right)}
$$

where $\mathbf{1}_{B\left(y^{i}+\gamma^{i}, r\right)}$ is the characteristic function. Observe that $p>0$ because $\beta>0$ and $q_{i}-\frac{\beta}{k}>q_{i}-Q \geq 0$. Next, $p$ is a pdf since

$$
\int_{\Omega} p d \mu=\beta+\sum_{i=1}^{k} \frac{q_{i}-\frac{\beta}{k}}{(2 r)^{m}} \int_{\Omega} \mathbf{1}_{B\left(y^{i}+\gamma^{i}, r\right)} d \mu=\sum_{i=1}^{k} q_{i}=1 .
$$

We check that $p \in \mathcal{M}^{+}(\Omega, b)$. Indeed, since $\gamma^{i}=0$ for all $i \geq 2$

$$
\begin{aligned}
E_{p}\left(\ell_{j}\right) & =\int_{x \in \Omega} \ell_{j}(x) \cdot p(x) d \mu(x) \\
& =\beta \int_{\Omega} x_{j} d \mu+\frac{1}{(2 r)^{m}} \sum_{i=1}^{k}\left(q_{i}-\frac{\beta}{k}\right) \int_{B\left(y^{i}+\gamma^{i}, r\right)} x_{j} d \mu \\
& =\frac{1}{2} \beta+\sum_{i=1}^{k}\left(q_{i}-\frac{\beta}{k}\right)\left(y_{j}^{i}+\gamma_{j}^{i}\right) \\
& =\frac{1}{2} \beta+\sum_{i=1}^{k} q_{i} y_{j}^{i}+\left(q_{1}-\frac{\beta}{k}\right) \gamma_{j}^{1}-\frac{\beta}{k} \sum_{i=1}^{k} y_{j}^{i} \\
& =\frac{1}{2} \beta+b_{j}+\beta\left(\frac{1}{k} \sum_{i=1}^{k} y_{j}^{i}-\frac{1}{2}\right)-\frac{\beta}{k} \sum_{i=1}^{k} y_{j}^{i} \\
& =b_{j} .
\end{aligned}
$$

Suppose that $f: \Omega \rightarrow \mathbb{R}$ is continuous. By the mean value theorem there exist $\xi^{i} \in B\left(y^{i}+\gamma^{i}, r\right)$ such that

$$
\begin{aligned}
E_{p}(f)=\int_{x \in \Omega} f(x) \cdot p(x) d \mu(x)=\beta \int_{\Omega} f d \mu+\sum_{i=1}^{k} \frac{1}{(2 r)^{m}}\left(q_{i}-\frac{\beta}{k}\right) \int_{B\left(y^{i}+\gamma^{i}, r\right)} f d \mu= \\
\beta \int_{\Omega} f d \mu+\sum_{i=1}^{k}\left(q_{i}-\frac{\beta}{k}\right) \cdot f\left(\xi^{i}\right)
\end{aligned}
$$

By our choice $\beta<\epsilon$ and $\left\|\xi^{i}-x^{i}\right\|_{\infty} \leq\left\|\xi^{i}-\left(y^{i}+\gamma^{i}\right)\right\|_{\infty}+\left\|y^{i}-x^{i}\right\|_{\infty}+\left\|\gamma^{i}\right\|_{\infty}<r+\frac{\delta}{3}+r<\delta$. This completes the proof.

Proof of Theorem 3.9. For $i=0, \ldots, m$ set

$$
\begin{aligned}
& \alpha_{0}=f\left(\rho_{0}\right) \\
& \alpha_{i}=f\left(\rho_{i}\right)-f\left(\rho_{i-1}\right)
\end{aligned}
$$

Observe that since by definition $b_{0}=1$ and $b_{m+1}=0$,

$$
\begin{aligned}
E_{q^{*}}(f \mid \mathcal{L}) & =\sum_{i=0}^{m} q^{*}\left(\rho_{i}\right) \cdot f\left(\rho_{i}\right) \\
& =\sum_{i=0}^{m}\left(b_{i}-b_{i+1}\right) f\left(\rho_{i}\right) \\
& =f\left(\rho_{0}\right)+\sum_{i=1}^{m} b_{i}\left(f\left(\rho_{i}\right)-f\left(\rho_{i-1}\right)\right) \\
& =\alpha_{0}+\sum_{i=1}^{m} b_{i} \alpha_{i}
\end{aligned}
$$

Let $\check{f}: \mathbb{R}^{m} \rightarrow \mathbb{R}$ be the affine function $\check{f}=\sum_{i=0}^{m} \alpha_{i} \ell_{i}$, namely

$$
\check{f}\left(x_{1}, \ldots, x_{m}\right)=\alpha_{0}+\sum_{i=1}^{m} \alpha_{i} x_{i}
$$

Claim 1: $\check{f}$ dominates $f$ on $\mathcal{L}$, namely $\check{f}(\lambda) \geq f(\lambda)$ for all $\lambda \in \mathcal{L}$.
Proof: By construction of $\check{f}$ and by the definition of $\rho_{j}$ in (11), for any $0 \leq j \leq m$

$$
\check{f}\left(\rho_{j}\right)=\sum_{i=0}^{j} \alpha_{i}=f\left(\rho_{j}\right)
$$

So $f$ and $\check{f}$ coincide on $\left\{\rho_{0}, \ldots, \rho_{m}\right\} \subseteq \mathcal{L}$. Assume the statement of the claim is false, namely there exists $\lambda \in \mathcal{L}$ such that $\check{f}(\lambda)<f(\lambda)$. Among all these $\lambda$ 's choose one which contains the longest leading run of $1, \ldots, 1$, namely $\lambda$ with the largest possible $k$ with $\rho_{k} \preceq \lambda$. Notice that $k<m$ because $\check{f}\left(\rho_{m}\right)=f\left(\rho_{m}\right)$ and $\rho_{m}$ is maximal in $\mathcal{L}$. In the lattice $\mathcal{L}$ set $\lambda^{\prime}=\lambda \vee \rho_{k+1}$. Notice that since $k$ is the largest such that $\rho_{k} \preceq \lambda$ it follows that $\lambda \wedge \rho_{k+1}=\rho_{k}$. Since $\left.f\right|_{\mathcal{L}}$ is supermodular

$$
f\left(\lambda \vee \rho_{k+1}\right)+f\left(\rho_{k}\right) \geq f(\lambda)+f\left(\rho_{k+1}\right)
$$

Since $\check{f}$ is affine, it is modular, hence

$$
\check{f}\left(\lambda \vee \rho_{k+1}\right)+\check{f}\left(\rho_{k}\right)=\check{f}(\lambda)+\check{f}\left(\rho_{k+1}\right)
$$

Subtract the first equation from the second, taking into account that $\check{f}\left(\rho_{i}\right)=f\left(\rho_{i}\right)$, to get

$$
\check{f}\left(\lambda \vee \rho_{k+1}\right)-f\left(\lambda \vee \rho_{k+1}\right) \leq \check{f}(\lambda)-f(\lambda)<0
$$

Thus $\check{f}\left(\lambda^{\prime}\right)<f\left(\lambda^{\prime}\right)$ and $\rho_{k+1} \preceq \lambda^{\prime}$, contradiction to the maximality of $k$. q.e.d
Claim 2: $\check{f}$ dominates $f$ on $\Omega=[0,1]^{m}$, namely $\check{f}(x) \geq f(x)$ for all $x \in \Omega$.
Proof: Since $\Omega=[0,1]^{m}$ is the convex hull of $\mathcal{L}=\{0,1\}^{m}$, every $x \in \Omega$ is a convex combination

$$
x=\sum_{\lambda \in \mathcal{L}} t_{\lambda} \cdot \lambda .
$$

Since $\check{f}$ is affine and $f$ is convex, Claim 1 implies that

$$
\check{f}(x)=\check{f}\left(\sum_{\lambda \in \mathcal{L}} t_{\lambda} \cdot \lambda\right)=\sum_{\lambda \in \mathcal{L}} t_{\lambda} \check{f}(\lambda) \geq \sum_{\lambda \in \mathcal{L}} t_{\lambda} f(\lambda) \geq f\left(\sum_{\lambda \in \mathcal{L}} t_{\lambda} \lambda\right)=f(x) .
$$

q.e.d

Claim 2 implies that for any $p \in \mathcal{M}(\Omega, \mathbf{b})$

$$
E_{p}(f) \leq E_{p}(\check{f})=E_{p}\left(\sum_{i=0}^{m} \alpha_{i} \ell_{i}\right)=\sum_{i=0}^{m} \alpha_{i} E_{p}\left(\ell_{i}\right)=\alpha_{0}+\sum_{i=1}^{m} \alpha_{i} b_{i}=E_{q^{*}}\left(\left.f\right|_{\mathcal{L}}\right) .
$$

However, $q^{*}$ extends to $\widetilde{q^{*}} \in \mathcal{M}(\Omega, b)$ and clearly $E_{\widetilde{q}^{*}}(f)=E_{q^{*}}\left(\left.f\right|_{\mathcal{L}}\right)$ so we get

$$
\max _{p \in \mathcal{M}(\Omega, \mathbf{b})} E_{p}(f)=E_{q^{*}}\left(\left.f\right|_{\mathcal{L}}\right)
$$

Notice that $E_{q^{*}}\left(\left.f\right|_{\mathcal{L}}\right)=\sum_{i=0}^{m} q^{*}\left(\rho_{i}\right) \cdot f\left(\rho_{i}\right)$. Lemma 3.10 applied to $q^{*} \in \mathcal{M}(\Omega, b)$ and the continuity of $f$ easily imply that there exist $\lambda \in \mathcal{M}^{+}(\Omega, b)$ such that $E_{\lambda}(f)$ is arbitrarily close to $E_{q^{*}}(f)$. If follows that $\sup _{\mathcal{M}^{+}(\Omega, \mathbf{b})} E_{p}(f)=E_{q^{*}}\left(\left.f\right|_{\mathcal{L}}\right)$.

## 4. $(L, b)$-stable probability measures

Main results. We consider $n$ iterations of the random vector $L=\left(\ell_{1}, \ldots, \ell_{m}\right)$ over $\Omega=$ $[0,1]^{m}$ from Section 3. We obtain a sequence $L^{1}, \ldots, L^{n}$ of random vectors in $\mathbb{R}^{m}$ with sample space $\Omega^{n}$. In fact

$$
L^{k}: \Omega^{n} \rightarrow \Omega \subseteq \mathbb{R}^{m}
$$

is the projection to the $k$ th factor. This is the universal process on $\Omega^{n}$, see Section 2.D. Denote by $\mathcal{M}^{+}(\Omega, n)$ the set of all non-degenerate probability measures on $\Omega^{n}$. We will identify these with the set of pdf's $p: \Omega^{n} \rightarrow \mathbb{R}$ which are non-vanishing a.e.

With the notation and terminology of Section 2.B we make the following definition.
Definition 4.1. Consider some $b \in \operatorname{int}(\Omega)=(0,1)^{m}$. A pdf $p \in \mathcal{M}^{+}(\Omega, n)$ is called $(L, b)-$ stable if for any $0 \leq k \leq n-1$

$$
E_{p}\left(L^{k+1} \mid L^{1}, \ldots, L^{k}\right)=b
$$

That is, the function $E_{p}\left(L^{k+1} \mid L^{1}, \ldots, L^{k}\right): \Omega^{k} \rightarrow \mathbb{R}^{m}$ is constant with value $b$ a.e. The set of all $(L, b)$-stable $p \in \mathcal{M}^{+}(\Omega, n)$ is denoted

$$
\mathcal{M}^{+}(\Omega, n, b)
$$

Definition 4.2. A function $f: \Omega^{n} \rightarrow \mathbb{R}$ is called fibrewise convex-supermodular if it is convexsupermodular at each fibre. Namely, for any $\tau \in \Omega^{k-1}$ and any $\theta \in \Omega^{n-k}$ the function $g: \Omega \rightarrow \mathbb{R}$ defined by $g: \omega \mapsto f(\tau, \omega, \theta)$ is convex-supermodular (Definition 3.5).

Example 4.3. For any $J=\left(j_{1}, \ldots, j_{n}\right) \in \mathcal{P}_{n}(m)$, see (5), let $\ell_{J}: \Omega^{n} \rightarrow \mathbb{R}$ denote the function $\ell_{J}=\ell_{j_{1}} \otimes \cdots \otimes \ell_{j_{n}}$, namely

$$
\ell_{J}:\left(\omega^{1}, \ldots, \omega^{n}\right) \mapsto \ell_{j_{1}}\left(\omega^{1}\right) \cdots \ell_{j_{n}}\left(\omega^{n}\right)
$$

Consider $f: \Omega^{n} \rightarrow \mathbb{R}$ of the form $f=h \circ g$ where $h: \mathbb{R} \rightarrow \mathbb{R}$ is convex and $g: \Omega^{n} \rightarrow \mathbb{R}$ is of the form

$$
g=\sum_{J \in \mathcal{P}_{n}(m)} a_{J} \cdot \ell_{J}
$$

where $a_{J} \geq 0$ for all $J \neq(0, \ldots, 0)$. Then $f$ is fibrewise convex-supermodular.
Proof: It is clear that if $\tau \in \Omega^{k-1}$ and $\theta \in \Omega^{n-k}$ then the restriction of $\ell_{J}$ to the fibre $\{\tau\} \times \Omega \times\{\theta\} \subseteq \Omega^{n}$ is equal to $\sum_{i=0}^{n} b_{i} \ell_{i}$ where $b_{i} \geq 0$ for all $i \geq 1$; In fact $b_{i}$ is the sum of all $a_{J}$ in which the $k$ th entry is equal to $i$. The result follows from Example 3.6(1).

The main results of this section are the following two theorems.
Theorem 4.4. Let $f: \Omega^{n} \rightarrow \mathbb{R}$ be a continuous function of the form $f=h \circ g$ in Example 4.3. Assume that $f \geq 0$. Suppose that $b \in \operatorname{int}(\Omega)=(0,1)^{m}$. Then

$$
\inf _{p \in \mathcal{M}^{+}(\Omega, n, b)} E_{p}\left(f \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right)=f\left(\omega^{1}, \ldots, \omega^{k}, b, \cdots, b\right) .
$$

Recall the upper supermodular vertex $q^{*}: \mathcal{L} \rightarrow \mathbb{R}$ from Definition 3.7. It extends to an atomic probability measure on $\Omega$ supported on $\mathcal{L}$ which we abusively denote $q^{*}$. Let

$$
q^{* \otimes k}
$$

be the obvious product probability measure on $\mathcal{L}^{k}$ as well as its extension to $\Omega^{k}$.
Set $q_{j}=q^{*}\left(\rho_{j}\right)$ for all $0 \leq j \leq m$. For any $J \in \mathcal{P}_{k}(m)$ denote

$$
\begin{aligned}
q_{J} & =\prod_{j \in J} q_{j} \\
\rho_{J} & =\left(\rho_{j_{1}}, \cdots, \rho_{j_{k}}\right) \in \Omega^{k}
\end{aligned}
$$

If $f: \Omega^{n} \rightarrow \mathbb{R}$ is measurable and $\left(\omega^{1}, \ldots, \omega^{k}\right) \in \Omega^{k}$, we obtain a measurable function $g: \Omega^{n-k} \rightarrow \mathbb{R}$ by $g(-)=f\left(\omega^{1}, \ldots, \omega^{k},-\right)$. If $Q$ is a probability measure on $\Omega^{n-k}$ we will write $E_{Q}\left(f\left(\omega^{1}, \ldots, \omega^{k},-\right)\right)$ for $E_{Q}(g)$.
Theorem 4.5. Let $f: \Omega^{n} \rightarrow \mathbb{R}$ be a continuous fibrewise convex-supermodular, $f \geq 0$. Then

$$
\begin{aligned}
& \sup _{p \in \mathcal{M}^{+}(\Omega, n, b)} E_{p}\left(f \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right)=E_{q^{*} \otimes n-k}\left(f\left(\omega^{1}, \ldots, \omega^{k},-\right)\right) \\
&=\sum_{J \in \mathcal{P}_{n-k}(m)} q_{J} \cdot f\left(\omega^{1}, \ldots, \omega^{k}, \rho_{J}\right) .
\end{aligned}
$$

In the remainder of this section we prove Theorems 4.4 and 4.5. Throughout we fix $b \in(0,1)^{m}$ and assume that it is non-increasing, namely $b_{1} \geq \cdots \geq b_{m}$.

Lemma 4.6. Consider some $p \in \mathcal{M}^{+}(\Omega, n, b)$. Suppose that $0 \leq k \leq n-1$ and consider $\omega^{1}, \ldots, \omega^{k} \in \Omega$. Set $p^{\prime}=p_{L^{k+1} \mid\left(L^{1}, \ldots, L^{k}\right)=\left(\omega^{1}, \ldots, \omega^{k}\right)}$; See Section 2.B. Then $p^{\prime} \in \mathcal{M}^{+}(\Omega, b)$.

Proof. First, $p^{\prime}$ is a pdf and $p^{\prime}>0$ a.e., see (9) in Section 2.B. Set $I=\{1, \ldots, k\}$ and $J=\{k+1\}$, subsets of $\{1, \ldots, n\}$. Write $L^{I}$ for the random vector $\left(L^{1}, \ldots, L^{k}\right)$ and $\omega^{I}=$
$\left(\omega^{1}, \ldots, \omega^{k}\right) \in \Omega^{k}$. We use Lemma 2.1 and the fact that $p \in \mathcal{M}^{+}(\Omega, n, b)$ to compute

$$
\begin{aligned}
E_{p^{\prime}}(L)=E_{p^{\prime}}\left(\omega \mapsto L^{k+1}(\omega)\right) & =E_{p^{\prime}}\left(\omega \mapsto E_{p}\left(L^{k+1} \mid L^{I \cup J}\right)\left(\omega^{I}, \omega\right)\right) \\
& =E_{p}\left(L^{k+1} \mid L^{I}=\omega^{I}\right)=E_{p}\left(L^{k+1} \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right)=b .
\end{aligned}
$$

By definition, then, $p^{\prime} \in \mathcal{M}^{+}(\Omega, b)$.
Lemma 4.7. Let $p^{1}, \ldots, p^{n} \in \mathcal{M}^{+}(\Omega, b)$. Then $p^{1} \otimes \cdots \otimes p^{n} \in \mathcal{M}^{+}(\Omega, n, b)$.
Proof. It is clear that $p^{1} \otimes \cdots \otimes p^{n}$ is a non degenerate pdf on $\Omega^{n}$. By Lemma 2.3(3) and since by definition $E_{p^{k+1}}(L)=b$,

$$
E_{p^{1} \otimes \cdots \otimes p^{n}}\left(L^{k+1} \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right)=E_{p^{k+1}}\left(L^{k+1}\right)=E_{p^{k+1}}(L)=b
$$

Since this holds for all $0 \leq k \leq n-1$, by definition $p^{1} \otimes \cdots \otimes p^{n} \in \mathcal{M}^{+}(\Omega, n, b)$.
Lemma 4.8 ( $n$-fold Approximation Lemma). Let $b \in(0,1)^{n}$ and consider $q \in \mathcal{M}(\Omega, b)$ with finite support $\left\{x^{1}, \ldots, x^{r}\right\}$ and set $q_{i}=q\left(\left\{x^{i}\right\}\right)$. Let $q^{\otimes k}$ denote the induced product measure on $\Omega^{k}$. Let $f: \Omega^{n} \rightarrow \mathbb{R}$ be continuous with $f \geq 0$. Then for any $\epsilon>0$ and any $0 \leq k \leq n$ there exists $P \in \mathcal{M}^{+}(\Omega, n, b)$ such that

$$
\left|E_{P}\left(f \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right)-E_{q^{\otimes(n-k)}}\left(f\left(\omega^{1}, \ldots, \omega^{k},-\right)\right)\right|<\epsilon
$$

for all $\omega^{1}, \ldots, \omega^{k} \in \Omega$.
Proof. Since $\Omega^{n}$ is compact, $f$ is bounded, i.e $\|f\|_{\infty}<\infty$. Since $f$ is uniformly continuous, we choose $\delta>0$ suitable for $\frac{\epsilon}{3 n}$. Apply Lemma 3.10 with $\delta$ and with $\frac{\epsilon}{3 n\|f\|_{\infty}}$ to obtain $p \in \mathcal{M}^{+}(\Omega, b)$ and $\beta<\frac{\epsilon}{3 n\|f\|_{\infty}}$ such that for any continuous $g: \Omega \rightarrow \mathbb{R}$ where $g \geq 0$,

$$
\begin{equation*}
E_{p}(g)=\beta \int_{\Omega} g d \mu+\sum_{i=1}^{r}\left(q_{i}-\frac{\beta}{r}\right) g\left(\xi^{i}\right) \tag{12}
\end{equation*}
$$

for some $\xi^{1}, \ldots, \xi^{r} \in \Omega$ such that $\left\|x^{i}-\xi^{i}\right\|_{\infty}<\delta$.
Set $[r]=\{1, \ldots, r\}$. For any $I=\left(i_{1}, \ldots, i_{n-k}\right) \in[r]^{n-k}$ set

$$
q_{I}=q_{i_{1}} \cdots q_{i_{n-k}} \quad \text { and } \quad x^{I}=\left(x^{i_{1}}, \ldots, x^{i_{n-k}}\right) \in \Omega^{n-k}
$$

and let $f_{I}: \Omega^{k} \rightarrow \mathbb{R}$ be the function

$$
f_{I}:\left(\omega^{1}, \ldots, \omega^{k}\right) \mapsto f\left(\omega^{1}, \ldots, \omega^{k}, x^{I}\right)
$$

Observe that

$$
\begin{equation*}
E_{q^{\otimes(n-k)}}\left(f\left(\omega^{1}, \ldots, \omega^{k},-\right)\right)=\sum_{I \in[r]^{n-k}} q_{I} \cdot f_{I}\left(\omega^{1}, \ldots, \omega^{k}\right) . \tag{13}
\end{equation*}
$$

It is clear that $\left\|f_{I}\right\|_{\infty} \leq\|f\|_{\infty}$ and that $f_{I}$ is uniformly continuous with the same $\delta$ suitable for $\frac{\epsilon}{3 n}$ as that for $f$. Recall $p \in \mathcal{M}^{+}(\Omega, b)$ that we chose at the start of the proof.
Claim: Consider some $0 \leq k<n$ and some $I \in[r]^{n-k-1}$. Then for any $\omega^{1}, \ldots, \omega^{k} \in \Omega$

$$
\left|E_{p}\left(\omega \mapsto f_{I}\left(\omega^{1}, \ldots, \omega^{k}, \omega\right)\right)-\sum_{i=1}^{r} q_{i} \cdot f_{I}\left(\omega^{1}, \ldots, \omega^{k}, x^{i}\right)\right|<\frac{\epsilon}{n} .
$$

Proof: Set $g(w)=f_{I}\left(\omega^{1}, \ldots, \omega^{k}, \omega\right)$. Clearly $g: \Omega \rightarrow \mathbb{R}$ is continuous and $\|g\|_{\infty} \leq\|f\|_{\infty}$. Moreover, it is uniformly continuous and clearly the same $\delta$ we chose for $f$ suitable for $\frac{\epsilon}{3 n}$ works for $g$. Since $E_{q}(g)=\sum_{i=1}^{r} q_{i} g\left(x^{i}\right)$ and since (12) holds

$$
\left|E_{p}(g)-E_{q}(g)\right| \leq \beta \int_{\Omega} g d \mu+\frac{\beta}{r} \sum_{i=1}^{r}\left|g\left(\xi^{i}\right)\right|+\sum_{i=1}^{r} q_{i}\left|g\left(\xi^{i}\right)-g\left(x^{i}\right)\right| .
$$

Since $\|g\|_{\infty} \leq\|f\|_{\infty}$ and $\beta<\frac{\epsilon}{3 n\|f\|_{\infty}}$ the first and second terms in this sum are less than $\frac{\epsilon}{3 n}$. Since $\left\|\xi^{i}-x^{i}\right\|_{\infty}<\delta$, the uniform continuity of $g$ implies that the same is true for the last term since $\sum_{i} q_{i}=1$. This completes the proof of the claim.

Set $P=p^{\otimes n}$. Then $P \in \mathcal{M}^{+}(\Omega, n, b)$ by Lemma 4.7. In light of (13), we complete the proof of the lemma by showing by downward induction on $0 \leq k \leq n$ that

$$
\begin{equation*}
\left|E_{p^{\otimes n}}\left(f \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right)-\sum_{I \in[r]^{n-k}} q_{I} \cdot f_{I}\left(\omega^{1}, \ldots, \omega^{k}\right)\right| \leq(n-k) \frac{\epsilon}{n} . \tag{14}
\end{equation*}
$$

The base of induction $k=n$ is a triviality since

$$
E_{p^{\otimes n}}\left(f \mid L^{1}, \ldots, L^{n}\right)\left(\omega^{1}, \ldots, \omega^{n}\right)=f\left(\omega^{1}, \ldots, \omega^{n}\right)
$$

and since $f_{I}=f$ and $q_{I}=1$ for the the only $I \in[r]^{0}$.
Assume inductively that (14) holds for some $1 \leq k \leq n$. Fix some $\omega^{1}, \ldots, \omega^{k-1} \in \Omega$. Lemmas 2.1 and 2.3(2) imply that for any

$$
E_{p}\left(\tau \mapsto E_{p^{\otimes n}}\left(f \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k-1}, \tau\right)\right)=E_{p^{\otimes n}}\left(f \mid L^{1}, \ldots, L^{k-1}\right)\left(\omega^{1}, \ldots, \omega^{k-1}\right)
$$

Viewing the left hand side of (14) with $\omega^{1}, \ldots, \omega^{k-1}$ fixed as a function of $\tau \in \Omega$, the linearity of expectation $E_{p}(-)$ implies

$$
\left|E_{p^{\otimes n}}\left(f \mid L^{1}, \ldots, L^{k-1}\right)\left(\omega^{1}, \ldots \omega^{k-1}\right)-\sum_{I \in[r]^{n-k}} q_{I} E_{p}\left(f_{I}\left(\omega^{1}, \ldots, \omega^{k-1},-\right)\right)\right|<(n-k) \frac{\epsilon}{n}
$$

Thanks to (13), in order to complete the induction step (to $k-1$ ) it remains to show by that

$$
\left|\sum_{I \in[r]^{n-k}} q_{I} \cdot E_{p}\left(f_{I}\left(\omega^{1}, \ldots, \omega^{k-1},-\right)\right)-\sum_{J \in[r]^{n-k+1}} q_{J} \cdot f_{J}\left(\omega^{1}, \ldots, \omega^{k-1}\right)\right|<\frac{\epsilon}{n} .
$$

Given $J=\left(i_{1}, \ldots, i_{n-k+1}\right) \in[r]^{n-k+1}$ set $I=\left(i_{2}, \ldots, i_{n-k+1}\right) \in[r]^{n-k}$ and observe that $q_{J}=q_{I} q_{i_{1}}$ and that $f_{J}\left(\omega^{1}, \ldots, \omega^{k-1}\right)=f_{I}\left(\omega^{1}, \ldots, \omega^{k-1}, x^{i_{1}}\right)$. By the Claim above

$$
\begin{aligned}
& \left|\sum_{I \in[r]^{n-k}} q_{I} E_{p}\left(f_{I}\left(\omega^{1}, \ldots, \omega^{k-1},-\right)\right)-\sum_{J \in[r]^{n-k+1}} q_{J} f_{J}\left(\omega^{1}, \ldots, \omega^{k-1}\right)\right|= \\
& \quad=\left|\sum_{I \in[r]^{n-k}} q_{I} E_{p}\left(f_{I}\left(\omega^{1}, \ldots, \omega^{k-1},-\right)\right)-\sum_{I \in\left[r r^{n-k}\right.} \sum_{i=1}^{r} q_{I} q_{i} f_{I}\left(\omega^{1}, \ldots, \omega^{k-1}, x^{i}\right)\right| \\
& \quad \leq \sum_{I \in[r]^{n-k}} q_{I} \cdot\left|E_{p}\left(f_{I}\left(\omega^{1}, \ldots, \omega^{k-1},-\right)\right)-\sum_{i=1}^{r} q_{i} f_{I}\left(\omega^{1}, \ldots, \omega^{k-1}, x^{i}\right)\right| \\
& \quad<\sum_{I \in[r]^{n-k-1}} q_{I} \cdot \frac{\epsilon}{n}=\frac{\epsilon}{n} .
\end{aligned}
$$

This completes the induction step.

Proof of Theorem 4.4. For any $J=\left(j_{1}, \ldots, j_{n}\right) \in \mathcal{P}_{n}(m)$ and any $\omega^{1}, \ldots, \widehat{\omega^{k}}, \ldots, \omega^{n} \in \Omega$ (meaning $\omega^{k}$ is omitted) we have

$$
\ell_{J}\left(\omega^{1}, \ldots, \omega^{k-1},-, \omega^{k+1}, \ldots, \omega^{n}\right)=\prod_{i \neq k} \ell_{j_{i}}\left(\omega^{i}\right) \cdot \ell_{j_{k}}(-) .
$$

Therefore, if $\lambda \in \mathcal{M}^{+}(\Omega, b)$ we get

$$
\begin{aligned}
E_{\lambda}\left(\ell_{J}\left(\omega^{1}, \ldots, \omega^{k-1},-, \omega^{k+1}, \ldots, \omega^{n}\right)\right)=E_{\lambda}\left(\ell_{j_{k}}\right) \cdot \prod_{i \neq k} \ell_{j_{i}}\left(\omega^{i}\right)=b_{j_{k}} \cdot \prod_{i \neq k} \ell_{j_{i}}\left(\omega^{i}\right)= \\
\ell_{j_{k}}(b) \cdot \prod_{i \neq k} \ell_{j_{i}}\left(\omega^{i}\right)=\ell_{J}\left(\omega^{1}, \ldots, \omega^{k-1}, b, \omega^{k+1}, \ldots, \omega^{n}\right) .
\end{aligned}
$$

We use downward induction on $0 \leq k \leq n$ to show that for any $p \in \mathcal{M}^{+}(\Omega, n, b)$

$$
E_{p}\left(f \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right) \geq f\left(\omega^{1}, \ldots, \omega^{k}, b, \ldots, b\right)
$$

almost everywhere. The base of induction $k=n$ is a triviality (and in fact, equality holds a.e). Assume the inequality holds for $k+1 \leq n$. Set $p^{\prime}=p_{L^{k+1} \mid L^{1}=\omega^{1}, \ldots, L^{k}=\omega^{k}}$. Then $p^{\prime} \in \mathcal{M}^{+}(\Omega, b)$ by Lemma 4.6. Lemma 2.1 and the induction hypothesis imply

$$
\begin{aligned}
E_{p}\left(f \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right) & =E_{p^{\prime}}\left(\omega \mapsto E_{p}\left(f \mid L^{1}, \ldots, L^{k+1}\right)\left(\omega^{1}, \ldots, \omega^{k}, \omega\right)\right) \\
& \geq E_{p^{\prime}}\left(\omega \mapsto f\left(\omega^{1}, \ldots, \omega^{k}, \omega, b, \ldots, b\right)\right.
\end{aligned}
$$

Since $f=h \circ g$ with $h$ convex and $g$ as in Example 4.3, Jensen's inequality allows us to continue the inequality

$$
\begin{aligned}
& \geq h\left(\sum_{J} a_{J} E_{p^{\prime}}\left(\ell_{J}\left(\omega^{1}, \ldots, \omega^{k},-, b, \ldots, b\right)\right)\right. \\
& =h\left(\sum_{J} a_{J} \ell_{J}\left(\omega^{1}, \ldots, \omega^{k}, b, \ldots, b\right)\right) \\
& =h\left(g\left(\omega^{1}, \ldots, \omega^{k}, b, \ldots, b\right)\right) \\
& =f\left(\omega^{1}, \ldots, \omega^{k}, b, \ldots, b\right)
\end{aligned}
$$

This completes the induction step.
We deduce that in the statement of the theorem the right hand side is a lower bound for the left hand side and it remains to show equality. Let $\nu$ be the probability measure on $\Omega$ supported on $\{b\}$, i.e $\nu(\{b\})=1$. It is clear that for any measurable function $g: \Omega^{k} \rightarrow \mathbb{R}$ we have $E_{\nu \otimes k}(g)=g(b, \ldots, b)$. By Lemma 4.8, for any $\epsilon>0$ there exists $P \in$ $\mathcal{M}^{+}(\Omega, n, b)$ such that $E_{P}\left(f \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right)$ is $\epsilon$-close to $E_{\nu \otimes(n-k)}\left(f\left(\omega^{1}, \ldots, \omega^{k},-\right)\right)=$ $f\left(\omega^{1}, \ldots, \omega^{k}, b, \ldots, b\right)$. This completes the proof.

Proof of Theorem 4.5. First, observe that

$$
\begin{equation*}
E_{q^{* \otimes(n-k)}}\left(f\left(\omega^{1}, \ldots, \omega^{k},-\right)\right)=\sum_{I \in \mathcal{P}_{n-k}(m)} q_{I} \cdot f\left(\omega^{1}, \ldots, \omega^{k}, \rho_{I}\right) . \tag{15}
\end{equation*}
$$

Next, we prove that for any $p \in \mathcal{M}^{+}(\Omega, n, b)$ and any $0 \leq k \leq n$

$$
\begin{equation*}
E_{p}\left(f \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right) \leq E_{\left(q^{*}\right)^{\otimes n-k}}\left(f\left(\omega^{1}, \ldots, \omega^{k},-\right)\right) . \tag{16}
\end{equation*}
$$

Fix some $p$ and use downward induction on $k$. The base of induction $k=n$ is a triviality since $E_{p}\left(f \mid L^{1}, \ldots, L^{n}\right)=f$ a.e. Assume inductively that (16) holds for $k+1 \leq n$. Set
$p^{\prime}=p_{L^{k+1} \mid\left(L^{1}, \ldots, L^{k}\right)=\left(\omega^{1}, \ldots, \omega^{k}\right)}$. Lemma 2.1 and the induction hypothesis together with (15) imply that

$$
\begin{aligned}
E_{p}\left(f \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right) & =E_{p^{\prime}}\left(\omega \mapsto E_{p}\left(f \mid L^{1}, \ldots, L^{k+1}\right)\left(\omega^{1}, \ldots, \omega^{k}, \omega\right)\right) \\
& \leq E_{p^{\prime}}\left(\omega \mapsto \sum_{I \in \mathcal{P}_{n-k-1}} q_{I} f\left(\omega^{1}, \ldots, \omega^{k}, \omega, \rho_{I}\right)\right) \\
& =\sum_{I \in \mathcal{P}_{n-k-1}(m)} q_{I} \cdot E_{p^{\prime}}\left(\omega \mapsto f\left(\omega^{1}, \ldots, \omega^{k}, \omega, \rho_{I}\right)\right) .
\end{aligned}
$$

By the assumption on $f$, each function $f\left(\omega^{1}, \ldots, \omega^{k},-, \rho_{I}\right)$ is convex-supermodular and continuous and. Lemma 4.6 and Theorem 3.9 allow us to continue the estimate

$$
\left.\leq \sum_{I \in \mathcal{P}_{n-k-1}(m)} q_{I} \cdot \sum_{j=0}^{m} q_{j} \cdot f\left(\omega^{1}, \ldots, \omega^{k}, \rho_{j}, \rho_{I}\right)\right)=\sum_{I \in \mathcal{P}_{n-k}(m)} q_{I} \cdot f\left(\omega^{1}, \ldots, \omega^{k}, \rho_{I}\right)
$$

Together with (15), this completes the induction step.
We deduce that for any $0 \leq k \leq n$ the right hand side in the statement of the theorem is an upper bound for the left hand side. By Lemma 4.8 there exist $P \in \mathcal{M}^{+}(\Omega, n, b)$ such that $E_{P}\left(f \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right)$ are arbitrarily close to $E_{q^{* \otimes(n-k)}}\left(f\left(\omega^{1}, \ldots, \omega^{k},-\right)\right.$. This completes the proof.

## 5. Proofs of the main results

In this section we prove the results in Section $\mathbb{1}$. We start by setting up a formal framework for the discrete-time continuous-binomial market model presented there.

We begin with the "one-step" process, namely description of the price jumps $\Psi_{i}$ where $0 \leq i \leq m$. By definition $\Psi_{0}=R$ and $\Psi_{i}$ are chosen at random from the interval $\left[D_{i}, U_{i}\right]$. By choosing a linear homeomorphisms $\left[D_{i}, U_{i}\right] \cong[0,1]$, a natural sample space for the probability space underlying a single step is $\Omega=[0,1]^{m}$ and

$$
\begin{aligned}
& \Psi_{i}\left(x_{1}, \ldots, x_{m}\right)=D_{i}+\left(U_{i}-D_{i}\right) x_{i}, \\
& \Psi_{0}\left(x_{1}, \ldots, x_{m}\right)=R
\end{aligned}
$$

With the notation of Section 3, for any $1 \leq i \leq m$

$$
\Psi_{i}=D_{i} \ell_{0}+\left(U_{i}-D_{i}\right) \ell_{i}
$$

We will write $\Psi: \Omega \rightarrow \mathbb{R}^{m+1}$ for the random vector

$$
\Psi=\left(\Psi_{0}, \ldots, \Psi_{m}\right)
$$

The natural sample space for the $n$-step model is $\Omega^{n}$. We obtain a process $\Psi^{1}, \ldots, \Psi^{n}$ of the price changes at time $k$ :

$$
\Psi^{k}: \Omega^{n} \xrightarrow{L^{k}} \Omega \xrightarrow{\Psi} \mathbb{R}^{m+1}
$$

where $L^{k}$ is the projection to the $k$-th factor and $L^{1}, \ldots, L^{k}$ form the universal process on $\Omega^{n}$, see Section 2.D. Thus,

$$
\Psi_{i}^{k}=D_{i}+\left(U_{i}-D_{i}\right) L_{i}^{k}
$$

where $L_{i}^{k}$ is the $i$ th component of $L^{k}: \Omega^{n} \rightarrow \Omega \subseteq \mathbb{R}^{m+1}$ and we observe that (since $\ell_{0}=$ $1: \Omega \rightarrow \mathbb{R}$ )

$$
L_{i}^{k}=\ell_{0}^{\otimes(k-1)} \otimes \ell_{i} \otimes \ell_{0}^{\otimes(n-k-1)}
$$

Recall that we assume that $0<D_{i}<R<U_{i}$ so in particular $\Psi_{i}^{k}>0$ for all $i$ and all $k$.

The prices of the assets form an $\mathbb{R}^{m+1}$-valued process

$$
S^{0}, \ldots, S^{n}: \Omega^{n} \rightarrow \mathbb{R}^{m+1}
$$

where $S^{k}=\left(S_{0}^{k}, \ldots, S_{m}^{k}\right)$ is the vector of prices of the assets at time $k$. It is assumed by the model that

$$
S_{i}^{k}>0 \quad \text { for all } 0 \leq i \leq m \text { and } 0 \leq k \leq n
$$

By construction of the model, the processes $S^{0}, \ldots, S^{n}$ and $\Psi^{1}, \ldots, \Psi^{n}$ satisfy the relation

$$
S_{i}^{k}=S_{i}^{k-1} \cdot \Psi_{i}^{k} \quad(0 \leq i \leq m \text { and } 1 \leq k \leq n) .
$$

It is therefore clear that for any $0 \leq k \leq n$

$$
S_{i}^{k}=S_{i}^{0} \cdot \Psi_{i}^{1} \cdots \Psi_{i}^{k}
$$

Recall the definition of $\mathcal{P}_{k}(m)$ from (5) in Section 1. For $J=\left(j_{1}, \ldots, j_{k}\right) \in \mathcal{P}_{k}(m)$ set $\ell_{J}=\ell_{j_{1}} \otimes \cdots \otimes \ell_{j_{k}}$. It follows that

$$
S_{i}^{k}=\sum_{J \in \mathcal{P}_{k}(m)} a_{J} \cdot L_{j_{1}}^{1} \cdots L_{j_{k}}^{k}=\sum_{J \in \mathcal{P}_{k}(m)} a_{J} \cdot \ell_{J} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{n-k \text { times }}
$$

for some $a_{J} \geq 0$.
Comment: In Section 1 the processes $\Psi^{k}$ and $S^{k}$ were denoted $\Psi(k)$ and $S(k)$.
The European basket in Section 1 is the function (random variable) $F: \Omega^{n} \rightarrow \mathbb{R}$

$$
F=(\underbrace{\sum_{i=0}^{m} c_{i} \cdot S_{i}^{n}-K}_{G})^{+}
$$

where $c_{i} \geq 0$ for $1 \leq i \leq m$ and $K>0$ is some number. Notice that

$$
G=\sum_{J \in \mathcal{P}_{n}(m)} a_{J} \ell_{J}
$$

where $a_{J} \geq 0$ for all $J \neq(0, \ldots, 0)$. It follows from Example 4.3 and since $h(x)=x^{+}$is continuous, convex and non-negative that

Proposition 5.1. $F$ is continuous and fibrewise convex-supermodular and $F \geq 0$.
Recall that a non-degenerate probability measure $p$ on $\Omega^{n}$ is called risk neutral if for any $k \geq 0$ and any $j \geq 1$ such that $k+j \leq n$

$$
E_{p}\left(S^{k+j} \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right)=R^{j} \cdot S^{k}\left(\omega^{1}, \ldots, \omega^{k}\right) .
$$

We denote the set of these probability measures by RN.
Proposition 5.2. $\mathrm{RN}=\mathcal{M}^{+}(\Omega, n, b)$ where $b=\left(b_{1}, \ldots, b_{m}\right)$ is defined in (2) in Section (1.
Proof. Since $S_{i}^{k+j}=S_{i}^{k} \cdot \Psi_{i}^{k+1} \cdots \Psi_{i}^{k+j}$ and since $S_{i}^{k}>0$, it is clear that the condition for $p$ being a risk neutral measure is equivalent to the condition

$$
E_{p}\left(\Psi_{i}^{k+1} \cdots \Psi_{i}^{k+j} \mid L^{1}, \ldots, L^{k}\right)=R^{j}
$$

(almost everywhere constant function $\Omega^{n-k} \rightarrow \mathbb{R}$ ). It is easily verified using induction and Lemmas 2.2 and 2.1 that this condition (for any $k, j$ such that $k+j \leq n$ ) is equivalent to the single step condition, namely

$$
E_{p}\left(\Psi_{i}^{k+1} \mid L^{1}, \ldots, L^{k}\right)=R
$$

for all $1 \leq k \leq n-1$. But $\Psi_{i}^{k+1}=D_{i}+\left(U_{i}-D_{i}\right) \cdot L_{i}^{k+1}$. So the condition above is equivalent to

$$
D_{i}+\left(U_{i}-D_{i}\right) \cdot E_{p}\left(L_{i}^{k+1} \mid L^{1}, \ldots, L^{k}\right)=R .
$$

Using the definition of $b_{1}, \ldots, b_{m}$ in (2), this is equivalent to $E_{p}\left(L_{i}^{k+1} \mid L^{1}, \ldots, L^{k}\right)=b_{i}$, and collecting these for all $1 \leq i \leq m$ we get

$$
E_{p}\left(L^{k+1} \mid L^{1}, \ldots, L^{k}\right)=b
$$

which by definition is the condition for $p \in \mathcal{M}^{+}(\Omega, n, b)$.
Proof of Theorem [1.1. By Proposition 5.1 $F$ is continuous convex-supermodular and $F \geq 0$. The interval $\left(\Gamma_{\min }(F, k), \Gamma_{\max }(F, k)\right)$ of the rational values of $F$ at some state of the world $\left(\omega^{1}, \ldots, \omega^{k}\right) \in \Omega^{k}$ is known to be the collection of numbers

$$
\left\{R^{k-n} \cdot E_{p}\left(F \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right)\right\}_{p \in \mathrm{RN}}
$$

Proposition 5.2 and Theorem 4.4 imply that

$$
\begin{aligned}
& R^{n-k} \cdot \Gamma_{\min }(F, k)\left(\omega^{1}, \ldots, \omega^{k}\right)=\inf _{p \in \operatorname{RN}} E_{p}\left(F \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right)= \\
& \inf _{p \in \mathcal{M}^{+}(\Omega, n, b)} E_{p}\left(F \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right)=F\left(\omega^{1}, \ldots, \omega^{k}, b, \ldots, b\right) .
\end{aligned}
$$

By definition of $b$, see (2), and since $\Psi_{i}=D_{i} \ell_{0}+\left(U_{i}-D_{i}\right) \ell_{i}$,

$$
\Psi_{i}(b)=R \quad \text { for all } 1 \leq i \leq m
$$

By definition, for any $1 \leq i \leq m$

$$
S_{i}^{n}\left(\omega^{1}, \ldots, \omega^{k}, b, \ldots, b\right)=S_{i}^{0} \cdot \Psi_{i}\left(\omega^{1}\right) \cdots \psi_{i}\left(\omega^{k}\right) \cdot \underbrace{\Psi_{i}(b) \cdots \Psi_{i}(b)}_{n-k \text { times }}=R^{n-k} S_{i}^{k}\left(\omega^{1}, \ldots, \omega^{k}\right) .
$$

Also, for $i=0$ we clearly get $S_{0}^{n}=S_{0}^{0} \cdot R^{n}=R^{n-k} S_{0}^{k}$. It follows that

$$
F\left(\omega^{1}, \ldots, \omega^{k}, b, \ldots, b\right)=\left(R^{n-k} \sum_{i=0}^{m} c_{i} S_{i}^{k}-K\right)^{+}\left(\omega^{1}, \ldots, \omega^{k}\right)
$$

This establishes the formula for $\Gamma_{\min }(F, k)$.
We note that the numbers $q_{i}$ defined in (3) and used in the statement of the theorem are equal to $q^{*}\left(\rho_{i}\right)$ of the upper supermodular vertex (Definition 3.7). Since by Proposition 5.1 the conditions of Theorem 4.5 hold, it follows that

$$
\begin{aligned}
R^{n-k} \cdot \Gamma_{\max }(F, k)\left(\omega^{1}, \ldots, \omega^{k}\right) & =\sup _{p \in \mathrm{RN}} E_{p}\left(F \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right) \\
& =\sup _{p \in \mathcal{M}^{+}(\Omega, n, b)} E_{p}\left(F \mid L^{1}, \ldots, L^{k}\right)\left(\omega^{1}, \ldots, \omega^{k}\right) \\
& =\sum_{J \in \mathcal{P}_{n-k}(m)} q_{J} \cdot F\left(\omega^{1}, \ldots, \omega^{k}, \rho_{j_{1}}, \ldots, \rho_{j_{n-k}}\right) .
\end{aligned}
$$

Since $\ell_{i}\left(\rho_{j}\right)=1$ if $i \leq j$ and $\ell_{i}\left(\rho_{j}\right)=0$ if $i>j$ it follows that $\Psi_{i}\left(\rho_{j}\right)=D_{i}+\left(U_{i}-D_{i}\right) \ell_{i}\left(\rho_{j}\right)=$ $\chi_{i}(j)$. Therefore, for any $J \in \mathcal{P}_{n-k}(m)$,

$$
\begin{aligned}
S_{i}^{n}\left(\omega^{1}, \ldots, \omega^{k}, \rho_{j_{1}}, \ldots, \rho_{j_{n-k}}\right)=S_{i}^{0} \cdot \Psi_{i}\left(\omega^{1}\right) \cdots \Psi_{i}\left(\omega^{k}\right) \cdot \Psi_{i}\left(\rho_{j_{1}}\right) \cdots & \Psi_{i}\left(\rho_{j_{n-k}}\right)= \\
& \chi_{i}(J) \cdot S_{i}^{k}\left(\omega^{1}, \ldots, \omega^{k}\right) .
\end{aligned}
$$

For $i=0$ we get, of course, $S_{0}^{n}=S_{0}^{0} \cdot R^{n}=S_{0}^{k} \cdot \chi_{0}(J)$. Substitution into the definition of $F$ we get

$$
F\left(\omega^{1}, \ldots, \omega^{k}, \rho_{j_{1}}, \ldots, \rho_{j_{n-k}}\right)=\left(\sum_{i=0}^{m} c_{i} \cdot \chi_{i}(J) \cdot S_{i}^{k}-K\right)^{+}\left(\omega^{1}, \ldots, \omega^{k}\right)
$$

This establishes the formula for $\Gamma_{\max }(F, k)$.
Proof of Theorem 1.3. Recall that $\Psi_{i}(b)=D_{i} \ell_{0}-\left(U_{i}-D_{i}\right) \ell_{i}(b)=D_{i}+\left(U_{i}-D_{i}\right) b_{i}=R$ for $1 \leq i \leq m$ and that $\Psi_{0}=R$ by definition. By Theorem 1.1
$\Gamma_{\min }(F, 0)=R^{-n} \cdot F(b, \ldots, b)=R^{-n}\left(\sum_{i=0}^{m} c_{i} S_{i}^{0} \cdot \Psi_{i}(b)^{n}-K\right)^{+}=R^{-n} \cdot\left(R^{n} \sum_{i=0}^{m} c_{i} S_{i}^{0}-K\right)^{+}$.
This is independent of $U_{i}, D_{i}$.
Set $b_{i}$ as in (22) and denote by $b_{i}(s)$ the values of $b_{i}$ in our market model with parameters $u_{i}(s)$ and $d_{i}(s)$. Notice that $d_{i}(s)<R$ and that $u_{i}(s)>R$ for all $0 \leq s<1$. Moreover, $d_{i}(0)=D_{i}$ and $u_{i}(0)=U_{i}$ and $\lim _{s \nearrow 1} d_{i}(s)=R$ and $\lim _{s \nearrow 1} u_{i}(s)=R$. One checks that

$$
b_{i}(s)=\frac{R-d_{i}(s)}{u_{i}(s)-d_{i}(s)}=b_{i} .
$$

Therefore the values of $q_{i}=b_{i}-b_{i+1}$ are independent of $s$. The values of $\chi_{i}(j)$ do depend on $s$ where $\chi_{i}(j)(s)=u_{i}(s)$ if $i \leq j$ and $\chi_{i}(j)(s)=d_{i}(s)$ if $i>j$. Thus, $\chi_{i}(j)(s)$ is a polynomial (of degree 1) in $s$. Moreover, $\lim _{s} \nearrow \chi_{i}(j)(s)=R$. By Theorem 1.1

$$
\varphi(s)=\Gamma_{\max }\left(F, 0 ; u_{i}(s), d_{i}(s)\right)=R^{-n} \sum_{J \in \mathcal{P}_{n}(m)} q_{J} \cdot\left(\sum_{i=0}^{m} c_{i} \cdot S_{i}^{0} \cdot \prod_{j \in J} \chi_{i}(j)(s)\right)^{+}
$$

So $\varphi$ is a continuous function of $s \in[0,1)$. Now, $\varphi(0)=\Gamma_{\max }\left(F, 0 ; U_{i}, D_{i}\right)$ since $d_{i}(0)=D_{i}$ and $u_{i}(0)=U_{i}$. Since $h: x \mapsto x^{+}$is continuous and $\sum_{j=0}^{m} q_{j}=1$ we get

$$
\begin{aligned}
\lim _{s \nearrow 1} \varphi(s) & =R^{-n} \sum_{J \in \mathcal{P}_{n}(m)} q_{J}\left(\sum_{i=0}^{m} c_{i} S_{i}^{0} \cdot \lim _{s \nearrow 1} \prod_{j \in J} \chi_{i}(j)(s)-K\right)^{+} \\
& =R^{-n} \sum_{J \in \mathcal{P}_{n}(m)} q_{J}\left(\sum_{i=0}^{m} c_{i} S_{i}^{0} \cdot R^{n}-K\right)^{+} \\
& =R^{-n}\left(\sum_{i=0}^{m} c_{i} S_{i}^{0} \cdot R^{n}-K\right)^{+} \\
& =\Gamma_{\min }\left(F, 0 ; U_{i}, D_{i}\right) .
\end{aligned}
$$

The "intermediate value" result in the theorem follows from the continuity of $\varphi(s)$.
In the next lemma we will consider processes on $\Omega$, see Section 2.D. Recall that the universal process is $L^{1}, \ldots, L^{n}$ where $L^{k}$ is the projection $\Omega^{n} \rightarrow \Omega$ to the $k$-th factor followed by the inclusion to $\mathbb{R}^{m}$.

Lemma 5.3. Fix some $n$. Let $q$ be a probability measure on $\Omega$ supported on $\left\{\rho_{0}, \ldots, \rho_{m}\right\}$, see (11). Consider $\mathbb{R}$-valued processes

$$
r^{0}, \ldots, r^{n-1}>0 \quad \text { and } \quad X^{0}, \ldots, X^{n}
$$

and $\mathbb{R}^{m}$-valued processes

$$
d^{0}, \ldots, d^{n-1} \quad \text { and } \quad \Delta^{0}, \ldots, \Delta^{n-1}>0
$$

Further, consider an $\mathbb{R}^{m+1}$-valued processes

$$
S^{0}, \ldots, S^{n} \quad \text { and } \quad \Psi^{1}, \ldots, \Psi^{n}
$$

with components $S^{k}=\left(S_{0}^{k}, \ldots, S_{m}^{k}\right)$ and $\Psi^{k}=\left(\Psi_{0}^{k}, \ldots, \Psi_{m}^{k}\right)$. Assume that
(1) $S_{i}^{k}=S_{i}^{k-1} \cdot \Psi_{i}^{k}$ for all $1 \leq k \leq n$ and all $0 \leq i \leq m$.
(2) $\Psi_{0}^{k+1}=r^{k}$ and $\Psi_{i}^{k+1}=d_{i}^{k}+\Delta_{i}^{k} \cdot L_{i}^{k+1}$ for any $0 \leq k \leq n-1$, and for any $\omega^{K}=$ $\left(\omega^{1}, \ldots, \omega^{k}\right) \in \Omega^{k}$,

$$
E_{q}\left(\omega \mapsto \Psi_{i}^{k+1}\left(\omega^{K}, \omega\right)\right)=r^{k}\left(\omega^{K}\right)
$$

(3) $X^{k}$ are fibrewise convex-supermodular, and for any $0 \leq k \leq n-1$ and any $\omega^{K} \in \Omega^{k}$

$$
E_{q}\left(\omega \mapsto X^{k+1}\left(\omega^{K}, \omega\right)\right)=r^{k}\left(\omega^{K}\right) \cdot X^{k}\left(\omega^{K}\right)
$$

Then among all $\mathbb{R}^{m+1}$-valued processes $\beta^{0}, \ldots, \beta^{n-1}$ there exists a process $\alpha^{0}, \ldots, \alpha^{n-1}$ which minimises the $\mathbb{R}$-valued random process

$$
\begin{equation*}
V_{\beta}^{k}=\sum_{i=0}^{m} \beta_{i}^{k} \cdot S_{i}^{k} \tag{17}
\end{equation*}
$$

subject to the condition

$$
\begin{equation*}
\sum_{i=0}^{m} \beta_{i}^{k} \cdot S_{i}^{k+1} \geq X^{k+1} \tag{18}
\end{equation*}
$$

Moreover, $V_{\alpha}^{k}=X^{k}$ and the value of $\alpha^{k}\left(\omega^{K}\right)$ at $\omega^{K} \in \Omega^{k}$ can be computed by

$$
\left[\begin{array}{c}
\alpha_{0}^{k}\left(\omega^{K}\right) \\
\alpha_{1}^{k}\left(\omega^{K}\right) \\
\vdots \\
\alpha_{m}^{k}\left(\omega^{K}\right)
\end{array}\right]=T\left(\omega^{K}\right)^{-1} \cdot M^{\prime}\left(\omega^{K}\right)^{-1} \cdot Q \cdot\left[\begin{array}{c}
X^{k+1}\left(\omega^{K} \rho_{0}\right) \\
X^{k+1}\left(\omega^{K} \rho_{1}\right) \\
\vdots \\
X^{k+1}\left(\omega^{K} \rho_{m}\right)
\end{array}\right]
$$

where $Q$ is the matrix in the statement of Theorem 1.2 and $T, M^{\prime}: \Omega^{k} \rightarrow \operatorname{Mat}_{(m+1) \times(m+1)}(\mathbb{R})$ are the $(m+1) \times(m+1)$ matrices

$$
\begin{aligned}
& T=\left[\begin{array}{ccccc}
r^{k} \cdot S_{0}^{k} & & & \\
& S_{1}^{k} & & \\
& & & \ddots & \\
& & & & S_{m}^{k}
\end{array}\right] \\
& M^{\prime}=\left[\begin{array}{c|cccc}
1 & d_{1}^{k} & d_{2}^{k} & \cdots & d_{m}^{k} \\
\hline 0 & \Delta_{1}^{k} & & & \\
\vdots & & \ddots & & \\
0 & & & & \Delta_{m}^{k}
\end{array}\right]
\end{aligned}
$$

Proof. We prove the lemma in a sequence of claims.
Claim 1: If a process $\beta^{0}, \ldots, \beta^{n-1}$ satisfies (18) then $V_{\beta}^{k} \geq X^{k}$ for all $0 \leq k \leq n-1$.

Proof: Choose some $k$ and some $\omega^{K} \in \Omega^{k}$. Then (18) becomes the following system of (infinitely many) inequalities in the unknowns $\beta_{i}^{k}\left(\omega^{K}\right)$

$$
\begin{equation*}
\sum_{i=0}^{m} \beta_{i}^{k}\left(\omega^{K}\right) \cdot S_{i}^{k+1}\left(\omega^{K}, \omega\right) \geq X^{k+1}\left(\omega^{K}, \omega\right), \quad(\omega \in \Omega) \tag{19}
\end{equation*}
$$

Let $\Phi(\omega)$ denote the left hand side of (19) and $Y(\omega)$ denote its right hand side. These are random variables with domain $\Omega$ so $E_{q}(\Phi) \geq E_{q}(Y)$. By the hypotheses on $X^{k}$

$$
E_{q}(Y)=E_{q}\left(\omega \mapsto X^{k+1}\left(\omega^{K} \omega\right)=r^{k}\left(\omega^{K}\right) \cdot X^{k}\left(\omega^{K}\right)\right.
$$

By the hypotheses on $S^{k}$ and $\Psi^{k}$

$$
\begin{aligned}
E_{q}(\Phi)=\sum_{i=0}^{m} \beta_{i}^{k}\left(\omega^{K}\right) \cdot S_{i}^{k}\left(\omega^{K}\right) \cdot E_{q}(\omega & \left.\mapsto \Psi_{i}^{k+1}\left(\omega^{K}, \omega\right)\right)= \\
& r^{k}\left(\omega^{K}\right) \cdot \sum_{i=0}^{m} \beta_{i}^{k}\left(\omega^{K}\right) \cdot S_{i}^{k}\left(\omega^{K}\right)=r^{k}\left(\omega^{K}\right) \cdot V_{\beta}^{k}\left(\omega^{K}\right) .
\end{aligned}
$$

Since $r^{k}>0$ it follows that $V_{\beta}^{k}\left(\omega^{K}\right) \geq X^{k}\left(\omega^{K}\right)$.
q.e.d

Consider some $0 \leq k \leq n-1$ and some $\omega^{K}=\left(\omega^{\prime} \ldots, \omega^{k}\right) \in \Omega^{k}$. We will show in Claim 5 below that the system of $m+1$ linear equations with $m+1$ unknowns $\alpha_{0}^{k}\left(\omega^{K}\right), \ldots, \alpha_{m}^{k}\left(\omega^{K}\right)$

$$
\begin{equation*}
\sum_{i=0}^{m} S_{i}^{k}\left(\omega^{K}\right) \Psi_{i}^{k+1}\left(\omega^{K}, \rho_{j}\right) \cdot \alpha_{i}^{k}\left(\omega^{K}\right)=X^{k+1}\left(\omega^{K}, \rho_{j}\right), \quad 0 \leq j \leq m \tag{20}
\end{equation*}
$$

has a unique solution given by the matrices $Q, M^{\prime}, T$ as in the statement of the lemma. Since $S^{k}>0$ and $d^{k}$ and $\Delta^{k}>0$ are measurable, we obtain measurable functions $\alpha^{k}: \Omega^{k} \rightarrow \mathbb{R}^{m+1}$ which form a process over $\Omega$

$$
\alpha^{0}, \ldots, \alpha^{n-1}
$$

We remark that (20) are the inequalities in (19) corresponding to $\omega=\rho_{0}, \ldots, \rho_{m}$ with inequalities turned into equalities.
Claim 2: Consider some $\omega^{K} \in \Omega^{k}$ where $0 \leq k \leq n-1$. Then $\alpha^{k}\left(\omega^{K}\right)$ solves the inequalities (19) for all $\lambda \in \mathcal{L} \subseteq \Omega$.

Proof: As in Claim 1, write $\Phi(\omega)$ for the left hand side of (19) and $Y(\omega)$ for the right. The claim is that $\Phi(\lambda) \geq Y(\lambda)$ for all $\lambda \in \mathcal{L}$. Assume this is false, namely $\Phi(\lambda)<Y(\lambda)$ for some $\lambda \in \mathcal{L}$. Among all these $\lambda$ 's choose one for which $j$ is maximal with $\rho_{j} \preceq \lambda$; see Definition 3.3 and the discussion below it. Clearly $j<m$ because by definition of $\alpha^{k}\left(\omega^{K}\right)$ we have $\Phi\left(\rho_{i}\right)=Y\left(\rho_{i}\right)$ for all $0 \leq i \leq m$ and because $\rho_{m} \in \mathcal{L}$ is maximal. Set $\lambda^{\prime}=\lambda \vee \rho_{j+1}$. By the choice of $\lambda$ we get $\lambda \wedge \rho_{j+1}=\rho_{j}$. Since $S_{i}^{k+1}\left(\omega^{K}, \omega\right)=S_{i}^{k}\left(\omega^{K}\right) \cdot \Psi_{i}^{k+1}(\omega)$ and since the assignment $\omega \mapsto \Psi_{i}^{k+1}\left(\omega^{K}, \omega\right)$ is an affine function on $\Omega$, it follows that $\Phi: \Omega \rightarrow \mathbb{R}$ is affine. Therefore

$$
\Phi\left(\lambda^{\prime}\right)+\Phi\left(\rho_{j}\right)=\Phi(\lambda)+\Phi\left(\rho_{j+1}\right)
$$

The assumption on $X^{k+1}$ implies that $\left.Y\right|_{\mathcal{L}}$ is supermodular, hence

$$
Y\left(\lambda^{\prime}\right)+Y\left(\rho_{j}\right) \geq Y(\lambda)+Y\left(\rho_{j+1}\right)
$$

Subtracting these inequalities, keeping in mind that by construction $\Phi\left(\rho_{i}\right)=Y\left(\rho_{i}\right)$, we get

$$
\Phi\left(\lambda^{\prime}\right)-Y\left(\lambda^{\prime}\right) \leq \Phi(\lambda)-Y(\lambda)<0
$$

Therefore $\Phi\left(\lambda^{\prime}\right)<Y\left(\lambda^{\prime}\right)$ and $\rho_{j+1} \preceq \lambda^{\prime}$. This contradicts the maximality of $j$. q.e.d Claim 3: $\alpha^{k}\left(\omega^{K}\right)$ solves the inequalities (19) for all $\omega \in \Omega$.

Proof: Since $\Omega$ is the convex hull of $\mathcal{L}$, any $\omega \in \Omega$ is a convex combination $\omega=\sum_{\lambda \in \mathcal{L}} t_{\lambda} \cdot \lambda$. The assumption on $X^{k+1}$ implies that $Y: \Omega \rightarrow \mathbb{R}$ is convex. Together with Claim 2 and since $\Phi$ is affine

$$
\Phi(\omega)=\Phi\left(\sum_{\lambda \in \mathcal{L}} t_{\lambda} \lambda\right)=\sum_{\lambda \in \mathcal{L}} t_{\lambda} \Phi(\lambda) \geq \sum_{\lambda \in \mathcal{L}} t_{\lambda} Y(\lambda) \geq Y\left(\sum_{\lambda \in \mathcal{L}} t_{\lambda} \lambda\right)=Y(\omega) .
$$

q.e.d

Claim 4: $V_{\alpha}^{k}=X^{k}$.
Proof: Denote $q_{j}=q\left(\rho_{j}\right)$. Consider some $\omega^{K} \in \Omega^{k}$. Equation (20) defining $\alpha^{k}\left(\omega^{K}\right)$ yields

$$
\begin{aligned}
r^{k}\left(\omega^{K}\right) \cdot X^{k}\left(\omega^{K}\right) & =E_{q}\left(\omega \mapsto X^{k+1}\left(\omega^{K}, \omega\right)\right) \\
& =\sum_{j=0}^{m} q_{j} X^{k+1}\left(\omega^{K}, \rho_{j}\right) \\
& =\sum_{j=0}^{m} \sum_{i=0}^{m} \alpha_{i}^{k}\left(\omega^{K}\right) S_{i}^{k}\left(\omega^{K}\right) \cdot q_{j} \Psi_{i}^{k+1}\left(\omega^{K}, \rho_{j}\right) \\
& =\sum_{i=0}^{m} \alpha_{i}^{k}\left(\omega^{K}\right) S_{i}^{k}\left(\omega^{K}\right) \cdot E_{q}\left(\omega \mapsto \Psi_{i}^{k+1}\left(\omega^{K}, \omega\right)\right) \\
& =r^{k}\left(\omega^{K}\right) \cdot V_{\alpha}^{k}\left(\omega^{K}\right) .
\end{aligned}
$$

Since $r^{k}>0$ it follows that $V_{\alpha}^{k}\left(\omega^{K}\right)=X^{k}\left(\omega^{K}\right)$. q.e.d

Claim 3 implies that $\alpha^{0}, \ldots, \alpha^{n-1}$ solve all the inequalities (19) and hence it solves the constraints (18). Claims 1 and 4 imply that $V_{\alpha}^{k} \leq V_{\beta}^{k}$ for all $k$ and all $\beta^{0}, \ldots, \beta^{n-1}$ that satisfy (18). To complete the proof of the lemma it only remains to prove:
Claim 5: The system of equations (20) has a unique solution given by the matrices $Q, T, M^{\prime}$ as in the statement of the lemma.

Proof: Consider $0 \leq k \leq n-1$ and $\omega^{K} \in \Omega^{k}$. For $0 \leq i, j \leq m$ set $\chi_{i}^{k+1}(j)=\Psi_{i}^{k+1}\left(\omega^{K}, \rho_{j}\right)$. Notice that by the hypotheses

$$
\chi_{0}^{k+1}(j)=r^{k}\left(\omega^{K}\right)
$$

For $1 \leq i \leq m$ observe that $L_{i}^{k+1}\left(\rho_{j}\right)=1$ if $i \leq j$ and $L_{i}^{k+1}\left(\rho_{j}\right)=0$ if $i>j$. Since $\Psi^{k+1}=d_{i}^{k}+\Delta_{i}^{k}\left(\omega^{K}\right) L_{i}^{k+1}$ we get

$$
\chi_{i}^{k}(j)= \begin{cases}d_{i}^{k}\left(\omega^{K}\right)+\Delta_{i}^{k}\left(\omega^{K}\right) & i \leq j \\ d_{i}^{k}\left(\omega^{K}\right) & i>j\end{cases}
$$

Since $S_{i}^{k+1}\left(\omega^{K}, \rho_{j}\right)=S_{i}^{k}\left(\omega^{K}\right) \cdot \Psi_{i}^{k+1}\left(\omega^{K}, \rho_{j}\right)$, the matrix representing the system (20) is

$$
\begin{aligned}
& M=\left[\begin{array}{cccc}
S_{0}^{k} \chi_{0}^{k+1}(0) & S_{1}^{k} \chi_{1}^{k+1}(0) & \cdots & S_{m}^{k} \chi_{m}^{k+1}(0) \\
S_{0}^{k} \chi_{0}^{k+1}(1) & S_{1}^{k} \chi_{1}^{k+1}(1) & \cdots & S_{m}^{k} \chi_{m}^{k+1}(1) \\
\vdots & & & \vdots \\
S_{0}^{k} \chi_{0}^{k+1}(m) & S_{1}^{k} \chi_{1}^{k+1}(m) & \cdots & S_{m}^{k} \chi_{m}^{k+1}(m)
\end{array}\right]= \\
& \underbrace{\left[\begin{array}{cccc}
1 & \chi_{1}^{k+1}(0) & \cdots & \chi_{m}^{k+1}(0) \\
1 & \chi_{1}^{k+1}(1) & \cdots & \chi_{m}^{k+1}(1) \\
\vdots & & & \vdots \\
1 & \chi_{1}^{k+1}(m) & \cdots & \chi_{m}^{k+1}(m)
\end{array}\right]}_{M^{\prime \prime}} \cdot \underbrace{\left[\begin{array}{llll}
r^{k} S_{0}^{k} & & & \\
& S_{1}^{k} & & \\
& & \ddots & \\
& & & S_{m}^{k}
\end{array}\right]}_{T}
\end{aligned}
$$

with all entries evaluated at $\omega^{K}$. Thus, $M, M^{\prime \prime}, T$ are functions $\Omega^{k} \rightarrow \operatorname{Mat}_{(m+1) \times(m+1)}(\mathbb{R})$. Oserve that for all $1 \leq i \leq m$ and $1 \leq j \leq m$

$$
\chi_{i}^{k+1}(j)-\chi_{i}^{k+1}(j-1)= \begin{cases}\Delta_{i}^{k+1}\left(\omega^{K}\right) & \text { if } i=j \\ 0 & \text { if } i \neq j\end{cases}
$$

It follows that

$$
Q \cdot M^{\prime \prime}=\left[\begin{array}{c|cccc}
1 & d_{1}^{k} & d_{2}^{k} & \cdots & d_{m}^{k} \\
\hline 0 & \Delta_{1}^{k} & & & \\
0 & & \Delta_{2}^{k} & & \\
\vdots & & & \ddots & \\
0 & & & & \Delta_{m}^{k}
\end{array}\right]
$$

with these matrices evaluated at $\omega^{K}$. We denote the latter matrix by $M^{\prime}$ and notice that it is invertible since $\Delta_{i}^{k}>0$. In particular $M\left(\omega^{K}\right)$ is invertible for any $\omega^{K} \in \Omega^{k}$ so (20) has a unique solution. Note that $M^{-1}=T^{-1} \cdot M^{\prime-1} \cdot Q$ is a measurable function $\Omega^{n} \rightarrow$ $\operatorname{Mat}_{(m+1) \times(m+1)}(\mathbb{R})$ and the solution of (20) is therefore the one given in the statement of the lemma.

Proof of Theorem 1.2. We apply Lemma 5.3 with the following data. The probability measure $q$ is the upper supervertex $q^{*}: \mathcal{L} \rightarrow \mathbb{R}$ (Definition 3.7) extended to a probability measure on $\Omega$. The processes $S^{0}, \ldots, S^{n}$ and $\Psi^{1}, \ldots, \Psi^{n}$ are the prices $S^{k}=\left(S_{0}^{k}, \ldots, S_{m}^{k}\right)$ of the assets and their price jumps $\Psi^{k}=\left(\Psi_{0}^{k}, \ldots, \Psi_{m}^{k}\right)$. The process $r^{0}, \ldots, r^{n-1}$ consists of the constant functions with value $R$. The processes $d^{0}, \ldots, d^{n-1}$ and $\Delta^{0}, \ldots, \Delta^{n-1}$ have components $d_{i}^{k}$ and $\Delta_{i}^{k}(1 \leq i \leq m)$ where $d_{i}^{k}$ is constant with value $D_{i}$ and $\Delta_{i}^{k}$ is constant with value $U_{i}-D_{i}$. The process $X^{0}, \ldots, X^{n}$ are the upper bound of the option's $F$ price at time $k$, namely $X^{k}=\Gamma_{\text {max }}(F, k)$.

We need to show that the conditions of the lemma are fulfilled. First, $S_{i}^{k}>0$ for all $k$. By construction of the model, $S_{i}^{k+1}=S_{i}^{k} \cdot \Psi_{i}^{k+1}$ and $\Psi_{0}^{k+1}=R=r^{k}$ and $\Psi_{i}^{k+1}=$ $D_{i}+\left(U_{i}-D_{i}\right) L_{i}^{k}=d_{i}^{k}+\Delta_{i}^{k} L_{i}^{k+1}$ for all $0 \leq k \leq n-1$. Also,

$$
E_{q}\left(\omega \mapsto \Psi_{i}^{k+1}\left(\omega^{K}, \omega\right)\right)=E_{q}\left(D_{i}+\left(U_{i}-D_{i}\right) \ell_{i}(\omega)\right)=D_{i}+\left(U_{i}-D_{i}\right) b_{i}=R=r^{k}\left(\omega^{K}\right)
$$

by construction of $q^{*}$ (Definition 3.7). By Theorem 1.1

$$
X^{k}\left(\omega^{K}\right)=R^{k-n} \sum_{J \in \mathcal{P}_{n-k}(m)} q_{J} \cdot F\left(\omega^{K}, \rho_{J}\right)
$$

Since $F$ is fibrewise convex-supermodular by Proposition $5.1 X^{k}$ is a linear combination with non-negative coefficients of fibrewise convex-supermodular functions, hence it is one as well. Finally, we check that

$$
\begin{aligned}
E_{q^{*}}\left(\omega \mapsto X^{k+1}\left(\omega^{K}, \omega\right)\right) & =E_{q^{*}}\left(\omega \mapsto R^{k+1-n} \sum_{J \in \mathcal{P}_{n-k-1}(m)} q_{J} \cdot F\left(\omega^{K}, \omega, \rho_{J}\right)\right) \\
& =R^{k+1-n} \sum_{j=0}^{m} q_{j} \sum_{J \in \mathcal{P}_{n-k-1}(m)} q_{J} F\left(\omega^{K}, \rho_{j}, \rho_{J}\right) \\
& =R^{k+1-n} \sum_{J \in \mathcal{P}_{n-k}(m)} q_{J} F\left(\omega^{K}, \rho_{J}\right) \\
& =R \cdot X^{k}\left(\omega^{K}\right) \\
& =r^{k}\left(\omega^{K}\right) \cdot X^{k}\left(\omega^{K}\right) .
\end{aligned}
$$

All the conditions of Lemma 5.3 are fulfilled and we obtain a process $\alpha^{0}, \ldots, \alpha^{n-1}$ which minimises

$$
V_{\alpha}^{k}=\sum_{i=0}^{m} \alpha_{i}^{k} S_{i}^{k}
$$

subject to the requirement that for all $0 \leq k \leq n-1$

$$
\sum_{i=0}^{m} \alpha_{i}^{k} S_{i}^{k+1} \geq X^{k+1}=\Gamma_{\max }(F, k+1)
$$

Thus, $\alpha(k)=\alpha^{k}$ is a minimum-cost maximal hedging strategy as required with the formulas for its value given in the statement of the theorem. It only remains to note that $Y_{t}(k)$ at the state of the world $\omega^{K} \in \Omega^{k}$ used in the statement of the theorem is precisely $X^{k+1}\left(\omega^{K}, \rho_{t}\right)$ because

$$
\begin{aligned}
Y_{t}(k)\left(\omega^{K}\right) & =R^{k+1-n} \sum_{J \in \mathcal{P}_{n-k-1}(m)} q_{J} \cdot\left(\sum_{i=0}^{m} c_{i} \chi_{i}(J) \chi_{i}(t) S_{i}^{k}\left(\omega^{K}\right)-K\right)^{+} \\
& =R^{k+1-n} \sum_{J \in \mathcal{P}_{n-k-1}(m)} q_{J} \cdot\left(\sum_{i=0}^{m} c_{i} \chi_{i}(J) \Psi_{i}\left(\rho_{t}\right) S_{i}^{k}\left(\omega^{K}\right)-K\right)^{+} \\
& =R^{k+1-n} \sum_{J \in \mathcal{P}_{n-k-1}(m)} q_{J} \cdot\left(\sum_{i=0}^{m} c_{i} \chi_{i}(J) S_{i}^{k}\left(\omega^{K}, \rho_{t}\right)-K\right)^{+} \\
& =\Gamma_{\max }(F, k+1)\left(\omega^{K}, \rho_{t}\right) \\
& =X^{k+1}\left(\omega^{K}, \rho_{t}\right) .
\end{aligned}
$$

## References

[1] William Feller. An Introduction to Probability Theory and Its Applications. Vol. I. John Wiley \& Sons, Inc., New York, N.Y., 1950.
[2] Jürgen Franke, Wolfgang Karl Härdle, and Christian Matthias Hafner. Statistics of financial markets. Universitext. Springer, Cham, 2019. An introduction, Fifth edition of [ MR2102757].
[3] Allan Gut. Probability: a graduate course. Springer Texts in Statistics. Springer, New York, second edition, 2013.
[4] Paul R. Halmos. Measure Theory. D. Van Nostrand Co., Inc., New York, N. Y., 1950.
[5] Jarek Kędra, Assaf Libman, and Victoria Steblovskaya. Pricing and hedging contingent claims in a multiasset binomial market. arXiv:2106.13283, 2021.
[6] Jarek Kędra, Assaf Libman, and Victoria Steblovskaya. Hedging of european type contingent claims in discrete time binomial market models. Preprint, 2022.
[7] L. Lovász. Submodular functions and convexity. In Mathematical programming: the state of the art (Bonn, 1982), pages 235-257. Springer, Berlin, 1983.
[8] Stanley Pliska. Introduction to Mathematical Finance - Discrete Time Models. Blackwell Publishing, 1997.
[9] H. L. Royden. Real analysis. Macmillan Publishing Company, New York, third edition, 1988.
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[^0]:    ${ }^{1}$ If $p>0$ a.e then $\int_{E} p d u>0$ for any $E$ with $\mu(E)>0$ is a standard fact. If $p=0$ on $E$ with $\mu(E)=0$ then $\nu(E)=\int_{E} p(x) d \mu(x)=0$ so $\mu$ is not absolutely continuous with respect to $\nu$.

