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ON RATIONAL NUMBERS WITH SMALL NUMERATORS AND DENOMINATORS IN MUSIC

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Abstract

Rational numbers with small numerators and denominators play a special role in music. For example, the frequency ratio $\frac{2}{1}$ corresponds to the interval of an octave, whilst the frequency ratios $\frac{3}{2}, \frac{5}{4}, \frac{6}{5}$ correspond to the intervals of a major triad. These have led to various tunings of the set of notes on a stave involving rational ratios, known as just intonation. The differences between these tunings are quite subtle. Here, by the process of generalisation, we explore some other ways of using rational numbers with small numerators and denominators in music. We include a collection of tunings of the notes on the stave obtained in this way, where the differences between the tunings are not so subtle.

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1. Introduction

We describe some theoretical musical notions concerning rational numbers with small numerators and denominators, and present a number of examples. We proceed by generalisation: we take properties of music written on the stave, and describe alternative situations in which those same properties emerge. Our aim is to create pieces and scales that are diverse and, to a greater or lesser degree, harmonious.

Example 1. Consider two notes played simultaneously on a stringed instrument, whose frequencies differ by a factor of 2. This situation has the property that the frequency of the upper note is double the frequency of the lower note. An analogous situation is found where we have a piece *P* played, and accompanied by *P* played twice at twice the speed. Our sound file [8] gives this in case *P* is Bach's two part invention no. 9, tuned as in [7, Example 2].

In Section 2, we derive rational approximations of equal temperament, analogous to those in the 12 tone equal temperament case. A special case is 31 tone equal temperament.

In Section 3, we introduce various scales, analogous to just intonation approximating 12 tone equal temperament. Just scales approximating 12 tone equal temperament are only subtly different from each other, for example, a major third could be given by a frequency ratio of $\frac{5}{4}$ or $\frac{81}{64}$. Most of the scales we present here have frequencies which depart significantly from the corresponding frequencies in 12 tone equal temperament.

In Section 4, we discuss rescaling the harmonics of our notes, to conform with some of the properties of Section 3.

In Section 5, we associate some theoretical notions in 31 tone equal temperament, analogous to such notions in 12 tone equal temperament, and illustrate with an example. Another example is given in Section 6.

We give some rescalings in 31 tone equal temperament in Section 7.

The prime 11 is of special interest, as it is so badly approximated in 12 tone equal temperament $(2^{\frac{42}{12}}/11 \sim 1.0285)$. We give some examples in Section 8.

Note that there are other fixed scales, featuring more than twelve tones, constructed with rational numbers, such as Partch's 43 tone scale [3] (some background concerning just intonation can also be found in the same reference). Our approach almost entirely refers back to music written on a stave.

2. Rational Approximations of Equal Temperament

In this section, we discuss using approximate relations between integers, under multiplication, to derive equally tempered scales in $\mathbb{R}$, with rational approximations.

Let p be a prime, and $\mathcal{P}$ the set of primes $\leq p$. Let $\delta > 0$ be a real number. Then we define a quasi-relation to be an expression $\rho = 2^{a_2}3^{a_3} \cdots p^{a_p}$, where $a_i \in \mathbb{Z}$ for $i \in \mathcal{P}$, and $1 < \rho \leq 1 + \delta$.

Suppose we are given quasi-relations $\rho_j = 2^{a_{2j}}3^{a_{3j}} \cdots p^{a_{pj}}$, where $1 \leq j \leq |\mathcal{P}| - 1$. Suppose the vectors $(a_{2j}, \ldots, a_{pj})$ are linearly independent, for $1 \leq j \leq |\mathcal{P}| - 1$. Solving the set of $|\mathcal{P}| - 1$ equations in $|\mathcal{P}|$ variables,

$$a_{2j}x_2 + a_{3j}x_3 + \cdots + a_{pj}x_p = 0, \quad 1 \leq j \leq |\mathcal{P}| - 1,$$

yields a family of solutions $(x_2, x_3, \ldots, x_p) = \lambda y$, where λ runs through elements of $\mathbb{Q}$, and $y \in \mathbb{Q}^{\mathcal{P}}$ is fixed. There consequently exist integral solutions, of which we fix one $z = (z_2, z_3, \ldots, z_p)$.

Let us assume $z_2 > 0$ and fix $c \in \mathbb{R}$. We take an equally tempered scale $\mathfrak{C}$, with frequencies given by $c.2^{\frac{n}{z_2}}$, as n runs through the elements of

$\mathbb{Z}$. If we call the interval given by the frequency ratio $2^{\frac{1}{z_2}}$ a *basic interval*, this scale has z_2 basic intervals to an octave. For $q \in \mathcal{P}$, this scale has an endomorphism e_q, given by multiplication by $2^{\frac{z_q}{z_2}}$, which we think of as an approximation to multiplication by q. In place of the quasi-relations ρ_j, we have the relations $r_j = 1$, where $r_j = e_2^{a_{2j}} e_3^{a_{3j}} \cdots e_p^{a_{pj}}$, for $1 \leq j \leq |\mathcal{P}| - 1$.

Let us remark that many relations between the endomorphisms e_q can be exposed, by multiplying the elements r_j together.

For example, let us take $p = 5$ and $\delta = \frac{1}{40}$. Consider the quasi-relations $2^{-4}.3^4.5^{-1}$ and $2^7.5^{-3}$. In $\mathbb{Z}^{\{2,3,5\}}$, we have associated linearly independent vectors $(-4, 4, -1)$ and $(7, 0, -3)$. The equations

$$-4x_2 + 4x_3 - x_5 = 0, \quad 7x_2 - 3x_5 = 0$$

have an integral solution $(12, 19, 28)$. If we take $c = 440$, then we recover the equally tempered scale $\mathfrak{E}_{12}$ commonly used on a piano, with 12 basic intervals to an octave. We have a map $\phi : \mathbb{Z}^{\{2,3,5\}} \to \mathbb{Z}$ given by $(y_2, y_3, y_5) \mapsto 12y_2 + 19y_3 + 28y_5$. Then, for small y_is, $2^{y_2}3^{y_3}5^{y_5}$ is a rational approximation of $2^{\frac{\phi(y_2, y_3, y_5)}{12}}$, which gives, for example, a rational approximation of $\{2^{\frac{n}{12}}, 0 \leq n \leq 11\}$ as

$$\left\{1, \frac{3^3.5}{2^7}, \frac{3^2}{2^3}, \frac{3.5^2}{2^6}, \frac{5}{2^2}, \frac{3^3.5^2}{2^9}, \frac{3^2.5}{2^5}, \frac{3}{2}, \frac{5^2}{2^4}, \frac{3^3}{2^4}, \frac{3^2.5^2}{2^7}, \frac{3.5}{2^3}\right\}.$$

This corresponds to a tuning of the notes of the stave to just intonation – see Example 2.

For another example, let us take $p = 7$, and $\delta = \frac{1}{80}$. Consider the quasi-relations $2^{-4}.3^4.5^{-1}$, $2.3^2.5^{-3}.7$ and $2^{-5}.3^{-1}.5^{-2}.7^4$. In $\mathbb{Z}^{\{2,3,5,7\}}$, we have associated linearly independent vectors $(-4, 4, -1, 0)$, $(1, 2, -3, 1)$ and $(-5, -1, -2, 4)$. Applying row reduction to the equations

$$-4x_2 + 4x_3 - x_5 = 0,\ x_2 + 2x_3 - 3x_5 + x_7 = 0,\ -5x_2 - x_3 - 2x_5 + 4x_7 = 0,$$

we obtain an integral solution $(31, 49, 72, 87)$. Taking z to be this element, and $c = 440$, we obtain an equally tempered scale $\mathfrak{E}_{31}$ with 31 basic intervals to an octave, cf. [4].

We have a map $\phi : \mathbb{Z}^{\{2,3,5,7\}} \to \mathbb{Z}$ given by $(y_2, y_3, y_5, y_7) \mapsto 31y_2 + 49y_3 + 72y_5 + 87y_7$. Then, for small y_is, $2^{y_2}3^{y_3}5^{y_5}7^{y_7}$ is a rational approximation of $2^{\frac{\phi(y_2, y_3, y_5, y_7)}{31}}$ which gives, for example, a rational approximation of $\{2^{\frac{n}{31}}, 0 \le n \le 30\}$ as

$$\Sigma = \left\{1, \frac{2^6}{3^2.7}, \frac{3.7}{4.5}, \frac{2^4}{3.5}, \frac{7.5}{2^5}, \frac{3^2}{2^3}, \frac{2^3}{7}, \frac{7}{2.3}, \frac{2.3}{5}, \frac{2^7}{3.5.7}, \frac{5}{2^2}, \right.$$
$$\frac{2^5}{5^2}, \frac{3.7}{2^4}, \frac{2^2}{3}, \frac{7^2}{2^2.3^2}, \frac{7}{5}, \frac{2.5}{7}, \frac{5.7}{2^3.3}, \frac{3}{2}, \frac{2^5}{3.7}, \frac{5^2}{2^4}, \frac{2^3}{5},$$
$$\left. \frac{3.5.7}{2^6}, \frac{5}{3}, \frac{2^2.3}{7}, \frac{7}{2^2}, \frac{2^4}{3^2}, \frac{2^6}{5.7}, \frac{3.5}{2^3}, \frac{2^3.5}{3.7}, \frac{3^2.7}{2^5} \right\}.$$

The approximations of the fifth and seventh harmonics in 31 tone equal temperament are rather more accurate than the approximations in 12 tone equal temperament, although the approximation of the third harmonic is less accurate ($2^{\frac{19}{12}}/3 \sim 0.9989$ vs $2^{\frac{49}{31}}/3 \sim 0.9970$, $2^{\frac{28}{12}}/5 \sim 1.0079$ vs $2^{\frac{72}{31}}/5 \sim 1.0005$, $2^{\frac{34}{12}}/7 \sim 1.0182$ vs $2^{\frac{87}{31}}/7 \sim 0.9994$).

3. Some Scales

We introduce some scales, which are given by functions $f : \{0, 1, 2, ..., N\} \to \mathbb{R}$, where the note i semitones above the note two octaves below middle C have frequency $f(i)F$ Hz (here, F is some constant). Our audio files record Bach's two part invention no. 9 played in these scales, without overtones [8].

Let r be a real number > 1. We define $a \bmod \langle r \rangle$ to be the real number in $[1, r)$ that is equal to a modulo $\langle r \rangle$, in the multiplicative group of real numbers. For example, $9 \bmod \langle 2 \rangle = \frac{9}{8}$.

We introduce scales that possess properties that are also possessed by scales obtained by tuning the notes of the stave to just intonation.

Example 2. A scale tuned in just intonation in this way is obtained as follows. Let S be the 12 element set $\{3^m 5^n \bmod\langle 2 \rangle, 0 \le m \le 3, 0 \le n \le 2\}$, ordered with respect to the standard ordering on $\mathbb{R}$. Consider a map which is order-preserving with respect to the product ordering on its domain, $f : \{0, 1, 2, 3, 4\} \times S \to \mathbb{R}$ given by $(n, s) \mapsto 2^n s$. If we identify $\{0, 1, 2, 3, 4\} \times S$ with the lexicographic ordering, with the ordered set $\{0, 1, 2, 3, ..., 59\}$ (with its standard ordering <), this gives us a map from $\{0, 1, 2, 3, ..., 59\}$ to $\mathbb{R}$ which determines our scale.

The above scale has the following properties:

- Its frequency ratios are given by rational numbers.
- Its frequency ratios generate a subgroup of $\mathbb{Q}^\times$ of small rank, with a set of generators of small numerator and denominator.
- It approximates a scale in equal temperament.

The term 'small' here is deliberately vague. The idea is that the smaller a natural number is, the closer it is to 1, the frequency ratio of a unison. For us, the prime 2 is certainly small, equal to the frequency ratio of an octave, whilst the prime 29 is not especially small.

Here are some more scales constructed in a similar way with some of the same properties:

Example 3. Let S be the 12 element set $\{2^m \bmod \langle 3\rangle, 0 \leq m \leq 11\}$, ordered with respect to the standard ordering on $\mathbb{R}$. Consider a map which is order-preserving with respect to the product ordering on its domain, $f : \{0, 1, 2, 3, 4\} \times S \to \mathbb{R}$ given by $(n, s) \mapsto 3^n s$. If we identify $\{0, 1, 2, 3, 4\} \times S$ with the lexicographic ordering, with the ordered set $\{0, 1, 2, 3, ..., 59\}$, this gives us a map from $\{0, 1, 2, 3, ..., 59\}$ to $\mathbb{R}$ which determines our scale.

Example 4. Let S be the 12 element set $\{5^m \bmod \langle 2\rangle, 0 \leq m \leq 11\}$, ordered with respect to the standard ordering on $\mathbb{R}$. Consider a map which is order-preserving with respect to the product ordering on its domain, $f : \{0, 1, 2, 3, 4\} \times S \to \mathbb{R}$ given by $(n, s) \mapsto 2^n s$. If we identify $\{0, 1, 2, 3, 4\} \times S$ with the lexicographic ordering, with the ordered set $\{0, 1, 2, 3, ..., 59\}$, this gives us a map from $\{0, 1, 2, 3, ..., 59\}$ to $\mathbb{R}$ which determines our scale.

Example 5. Let S be the 12 element set $\{7^{-m} \bmod \langle 2\rangle, 0 \leq m \leq 11\}$, ordered with respect to the standard ordering on $\mathbb{R}$. Consider a map which is order-preserving with respect to the product ordering on its domain, $f : \{0, 1, 2, 3, 4\} \times S \to \mathbb{R}$ given by $(n, s) \mapsto 2^n s$. If we identify $\{0, 1, 2, 3, 4\} \times S$ with the lexicographic ordering, with the ordered set $\{0, 1, 2, 3, ..., 59\}$, this gives us a map from $\{0, 1, 2, 3, ..., 59\}$ to $\mathbb{R}$ which determines our scale.

Example 6. Let S be the 16 element set

$$\left\{\frac{n}{3},\ 3 \leq n \leq 14\right\} \cup \left\{\frac{n}{2},\ 2 \leq n \leq 9\right\},$$

ordered with respect to the standard ordering on $\mathbb{R}$. Consider a map which is order-preserving with respect to the product ordering on its domain, $f : \{0, 1, 2, 3\} \times S \to \mathbb{R}$ given by $(n, s) \mapsto 5^n s$. If we identify $\{0, 1, 2, 3\} \times S$ with the lexicographic ordering, with the ordered set $\{0, 1, 2, 3, ..., 63\}$, this gives us a map from $\{0, 1, 2, 3, ..., 63\}$ to $\mathbb{R}$ which determines our scale.

Example 7. Let S be the 11 element set $\left\{\frac{n}{11},\ 11 \leq n \leq 21\right\}$, ordered with respect to the standard ordering on $\mathbb{R}$. Consider a map which is order-preserving with respect to the product ordering on its domain, $f : \{0, 1, 2, 3, 4\} \times S \to \mathbb{R}$ given by $(n, s) \mapsto 2^n s$. If we identify $\{0, 1, 2, 3, 4\} \times S$ with the lexicographic ordering, with the ordered set $\{0, 1, 2, 3, ..., 54\}$, this gives us a map from $\{0, 1, 2, 3, ..., 54\}$ to $\mathbb{R}$ which determines our scale.

Example 8. Let S be the 12 element set $\{3^n \bmod \langle 2\rangle, -5 \leq n \leq 6\}$, ordered with respect to the standard ordering on $\mathbb{R}$. Consider a map which is order-preserving with respect to the product ordering on its domain, $f : \{0, 1, 2, 3, 4\} \times S \to \mathbb{R}$ given by $(n, s) \mapsto 2^n s$. If we identify $\{0, 1, 2, 3, 4\} \times S$ with the lexicographic ordering, with the ordered set $\{0, 1, 2, 3, ..., 59\}$, we obtain a sequence of 60 real numbers. Reversing this sequence gives us a map from $\{0, 1, 2, 3, ..., 59\}$ to $\mathbb{R}$ which determines our scale.

Example 9. Let S be the 12 element set $\left\{\frac{n}{3},\ 3 \leq n \leq 14\right\}$, ordered with respect to the standard ordering on $\mathbb{R}$. Consider a map which is order-preserving with respect to the product ordering on its domain, $f : \{0, 1\} \times S \to \mathbb{R}$ given by $(n, s) \mapsto 5^n s$. If we identify $\{0, 1\} \times S$ with the lexicographic ordering, with the ordered set $\{0, 1, 2, 3, ..., 23\}$, this gives us a map from $\{0, 1, 2, 3, ..., 23\}$ to $\mathbb{R}$. Concatenating this sequence with the single term sequence (5^2) and then the reverse 24 term sequence gives us a map from $\{0, 1, 2, 3, ..., 48\}$ to $\mathbb{R}$ which determines our scale.

Example 10. Let S be the 24 element set $\{n^2, 2 \le n \le 25\}$, ordered with respect to the standard ordering on $\mathbb{R}$. If we identify S with its ordering, with the ordered set $\{0, 1, 2, 3, ..., 23\}$, this gives us a map from $\{0, 1, 2, 3, ..., 23\}$ to $\mathbb{R}$. Concatenating this sequence with the single term sequence (26^2) and then the reverse 24 term sequence gives us a map from $\{0, 1, 2, 3, ..., 48\}$ to $\mathbb{R}$ which determines our scale.

Example 11. Let S be the 11 element set $\{3^m 5^n \bmod \langle 2\rangle, 0 \le m \le 3, 0 \le n \le 2\} \backslash \{3^3 5^2\}$, ordered with respect to the standard ordering on $\mathbb{R}$. Consider a map which is order-preserving with respect to the product ordering on its domain, $f : \{0, 1, 2, 3, 4\} \times S \to \mathbb{R}$ given by $(n, s) \mapsto 2^n s$. If we identify $\{0, 1, 2, 3, 4\} \times S$ with the lexicographic ordering, with the ordered set $\{0, 1, 2, 3, ..., 54\}$, this gives us a map from $\{0, 1, 2, 3, ..., 54\}$ to $\mathbb{R}$ which determines our scale.

Example 12. Consider a map which is order-preserving with respect to the product ordering on its domain, $f : \{1, 2, 3, 4, 5, 6, 7\} \times \{1, 2, 3, 4, 5, 6, 7\} \to \mathbb{R}$ given by $(n, s) \mapsto ns$. If we identify $\{1, 2, 3, 4, 5, 6, 7\}^2$ with the lexicographic ordering, with the ordered set $\{0, 1, 2, 3, ..., 48\}$, this gives us a map from $\{0, 1, 2, 3, ..., 48\}$ to $\mathbb{R}$ which determines our scale.

Example 13. Consider a map which is order-preserving with respect to the product ordering on its domain, $f : \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$\times \{1, 2, 3, 4, 5\} \to \mathbb{R}$ given by $(n, s) \mapsto ns$. If we identify

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \times \{1, 2, 3, 4, 5\}$$

with the lexicographic ordering, with the ordered set $\{0, 1, 2, 3, ..., 59\}$, this gives us a map from $\{0, 1, 2, 3, ..., 59\}$ to $\mathbb{R}$ which determines our scale.

Example 14. Let S be the 8 element set $\{3^m \bmod \langle 2\rangle, -2 \le m \le 5\}$, and let T be the 12 element set $(S \bigcup 2S) \bigcap [1, 3)$, ordered with respect to the standard ordering on $\mathbb{R}$. Consider a map which is order-preserving with respect to the product ordering on its domain, $f : \{0, 1, 2, 3, 4\} \times T \to \mathbb{R}$ given by $(n, s) \mapsto 3^n s$. If we identify $\{0, 1, 2, 3, 4\} \times S$ with the lexicographic ordering, with the ordered set $\{0, 1, 2, 3, ..., 59\}$, this gives us a map from $\{0, 1, 2, 3, ..., 59\}$ to $\mathbb{R}$ which determines our scale.

Example 15. Let S be the 8 element set $\{3^m \bmod \langle 2\rangle, -2 \le m \le 5\}$, and let T be the 12 element set $(S \bigcup 2S) \bigcap [1, 3)$, ordered with respect to the standard ordering on $\mathbb{R}$. Consider a map which is order-preserving with respect to the product ordering on its domain $f : \{0, 1, 2, 3, 4\} \times T \to \mathbb{R}$, given by $(n, s) \mapsto 2^n s$. If we identify $\{0, 1, 2, 3, 4\} \times S$ with the lexicographic ordering, with the ordered set $\{0, 1, 2, 3, ..., 59\}$, this gives us a map from $\{0, 1, 2, 3, ..., 59\}$ to $\mathbb{R}$ which determines our scale.

Example 16. Let $S = \Sigma$ be the 31 element set defined in Section 2, ordered with respect to the standard ordering on $\mathbb{R}$. Consider a map which is order-preserving with respect to the product ordering on its domain, $f : \{0, 1\} \times S \to \mathbb{R}$ given by $(n, s) \mapsto 2^n s$. If we identify $\{0, 1\} \times S$ with the lexicographic ordering, with the ordered set $\{0, 1, 2, 3, ..., 61\}$, we obtain a map from $\{0, 1, 2, 3, ..., 61\}$ to $\mathbb{R}$ which determines our scale.

Example 17. Let S be the 12 element set $\{1\} \bigcup \left\{\frac{n+1}{n}, 2 \le n \le 12\right\}$, ordered with respect to the standard ordering on $\mathbb{R}$. Consider a map which is order-preserving with respect to the product ordering on its domain, $f : \{0, 1, 2, 3, 4\} \times S \to \mathbb{R}$ given by $(n, s) \mapsto 2^n s$. If we identify $\{0, 1, 2, 3, 4\} \times S$ with the lexicographic ordering, with the ordered set $\{0, 1, 2, 3, ..., 59\}$, this gives us a map from $\{0, 1, 2, 3, ..., 59\}$ to $\mathbb{R}$ which determines our scale.

To end this section, for contrast, we record an example of a scale which does not approximate a scale in equal temperament, whose frequency ratios are given by rational numbers, and where the 50 element set of frequencies generates a subgroup of $\mathbb{Q}$ of rank 50.

Example 18. Let S be the 50 element set consisting of prime numbers, running from 31 to 281, ordered with respect to the standard ordering on $\mathbb{R}$. Identifying this with the ordered set $\{0, 1, 2, 3, ..., 49\}$, we obtain a map from $\{0, 1, 2, 3, ..., 49\}$ to $\mathbb{R}$ which determines our scale.

4. Rescaling Harmonics

The scales of the preceding section were fixed, as the piece progressed. In this section, the tunings potentially alter as the piece progresses. More examples of this phenomenon are to be found in the paper of Stange et al. [5].

In the preceding section, we have altered the notes of our Bach invention by fixing frequencies for the notes on the stave. An alternative approach, which we have used in a previous work, is to rescale the harmonics of our notes, and glue the pieces together along consonances, which we consider to be coincidences between harmonics of the original piece. This works roughly as follows: (for more details, see [6]) choose positive real numbers ζ_2, ζ_3 and ζ_5 to correspond to frequency ratios for the second, third and fifth harmonics, respectively (the frequency ratios for the fourth and sixth harmonics are then ζ_2^2 and $\zeta_2.\zeta_3$, respectively). We then have an associated interpretation of our piece with harmonics given by these frequency ratios, and certain consonances between notes corresponding to consonances in the original piece. When we do this, the pitch drift from the first C of the Bach invention to the last C of the piece at the same point on the stave is $\zeta_2^{-18}\zeta_3^{32}\zeta_5^{-14}$.

We wish to present some examples when the ζ_is are chosen so that $\langle\zeta_2, \zeta_3, \zeta_5\rangle$ has rank ≤ 2 (in imitation of the condition on scales in the preceding section, where frequencies were chosen to generate a subgroup of $\mathbb{Q}^\times$ of 'small' rank), and the ζ_is are given by fractions with small numerator and denominator (also in imitation of a condition in the preceding section).

To obtain a rank 2 group $\langle\zeta_2, \zeta_3, \zeta_5\rangle$, we take a relation between the ζ_is, which we insist takes some simple form. To help the piece be audible, we insist the pitch drift p from the first C of the piece to the last C of the piece at the same point on the stave is approximately equal to 1.

Example 19. For our first example (rank 2), we take $\zeta_5 = \zeta_2^2$ to be our relation. If we write $\zeta_3 = \zeta_2^a$, then the pitch drift constraint $p = 1$ gives us $a = \frac{23}{16}$. If we take $\zeta_2 = 5$, then $\zeta_3 \approx 10$ and $\zeta_5 = 25$. So, we finally fix $\zeta_2 = 5$, $\zeta_3 = 10$ and $\zeta_5 = 25$.

For our second example (rank 1), we take $(\zeta_2, \zeta_3, \zeta_5) = (1, 2, 4)$.

Here, we set an initial frequency $F = 528$ for the second example, and $F = 264$ for the first.

5. 31 **Tone Equal Temperament**

*Ostinati***.** The quasi-relation $2^{-4}.3^4.5^{-1}$ corresponds to the relation $e_2^{-4}.e_3^4.e_5^{-1} = 1$, in 12 tone equal temperament, where e_2 is multiplication by

2, e_3 is multiplication by $2^{\frac{19}{12}}$, and e_5 is multiplication by $2^{\frac{28}{12}}$. Inverting this relation, ignoring the octaves, and applying successive factors of the relation to the note C gives us a sequence C, E, A, D, G, C. Concatenating this with itself many times gives an ostinato which is a manifestation of the fifties progression:

$$C \xrightarrow{e_5} E \xrightarrow{e_3^{-1}} A \xrightarrow{e_3^{-1}} D \xrightarrow{e_3^{-1}}$$

$$G \xrightarrow{e_3^{-1}} C \xrightarrow{e_5} E \xrightarrow{e_3^{-1}} A \xrightarrow{e_3^{-1}} \cdots.$$

In general, when we have a relation $r = e_2^{a_2} e_3^{a_3} \cdots e_p^{a_p} = 1$ between e_is, we have associated ostinati comprising sequences of successively consonant notes, obtained by applying factors of r_j in sequence to a single note. Here, successive factors in the sequence differ by $e_i^{\pm 1}$.

For example, we have seen the quasi-relation $2^{-5}.3^{-1}.5^{-2}.7^4$ lifts to a relation in 31 tone equal temperament, whose inverse is $e_2^5 e_3 e_5^2 e_7^{-4}$. This gives us a sequence

$$F_0,\ e_3(F_0),\ e_3e_5(F_0),\ e_2e_3e_5(F_0),\ e_2e_3e_5^2(F_0),\ e_2^2e_3e_5^2(F_0),$$

$$e_2^2e_3e_5^2e_7^{-1}(F_0),\ e_2^3e_3e_5^2e_7^{-1}(F_0),\ e_2^3e_3e_5^2e_7^{-2}(F_0),\ e_2^4e_3e_5^2e_7^{-2}(F_0),$$

$$e_2^4e_3e_5^2e_7^{-3}(F_0),\ e_2^5e_3e_5^2e_7^{-3}(F_0),\ e_2^5e_3e_5^2e_7^{-4}(F_0) = F_0,$$

which, concatenated with itself a number of times gives a repeated ostinato O of successively consonant notes.

*Scales***.** The C major scale is obtained by applying a sequence of six successive consonant intervals (perfect fifths) to an initial note:

$$F,\ C,\ G,\ D,\ A,\ E,\ B.$$

The sequence of five notes C, G, D, A, E forms our fifties progression, in reverse. Any two sequences of d successive notes in this sequence differ by a transposition.

Sequences of endomorphisms e_i with $i > 2$ correspond to sequences of successive consonant intervals, which can be chosen to contain an ostinato, as well as transpositions. Note here we disregard e_2 as we identify notes in our scale up to octave equivalence. For example, extending the ostinato of the preceding subsection gives us the scale Θ:

$$F_0,\ e_3(F_0),\ e_3e_5(F_0),\ e_3e_5^2(F_0),\ e_3e_5^2e_7^{-1}(F_0),\ e_3e_5^2e_7^{-2}(F_0),\ e_3e_5^2e_7^{-3}(F_0),$$

$$e_3^2e_5^2e_7^{-3}(F_0),\ e_3^2e_5^3e_7^{-3}(F_0),\ e_3^2e_5^4e_7^{-3}(F_0),\ e_3^2e_5^4e_7^{-4}(F_0).$$

The first five notes of Θ, and the last five notes of Θ differ by a transposition.

There is a natural correspondence between scales of n notes constructed in this way: if we order such a scale by $<$ within the interval $[1, 2)$, its elements are in ordered correspondence with the elements of the totally ordered set $\{1, 2, 3, ..., n\}$.

*Distribution***.** We can use the notes in an existing piece written on the stave to determine the distribution of notes in a new piece in one of the aforementioned scales.

Example 20. Consider the set $\{1, 2, 3, 4, ..., 44\}$, which is in bijection with the set of half bars of Bach's two part invention no. 1 [1] (the number n corresponds to the nth half bar). We have a natural bijection between the 11 notes

$$B^\flat,\ F,\ C,\ G,\ D,\ A,\ E,\ B,\ F^\sharp,\ C^\sharp,\ G^\sharp$$

of the invention and the notes of Θ, respecting the order in which these are given above. To each n with $1 \le n \le 44$, we have a subset $\theta(n) \subset \Theta$ consisting of the notes of the nth half bar of our invention, translated via our natural bijection to a subset of Θ.

Our piece consists of the ostinato O, with notes in Θ, repeated four times, where notes have 2 seconds duration, along with an accompaniment. The first $\frac{6}{3+|\theta(n)|}$ seconds of our accompaniment to the nth note of the repeated ostinato consist of the following note of the ostinato, transposed to the octave between $2^3 F_0$ and $2^4 F_0$ if n is odd and transposed to the octave between $2^5 F_0$ and $2^6 F_0$ if n is even. The remaining $\frac{2|\theta(n)|}{3+|\theta(n)|}$ seconds of our accompaniment to the nth note of the repeated ostinato consists of the elements of $\theta(n)$, transposed to the octave between $2^4 F_0$ and $2^5 F_0$, and played with duration $\frac{2}{3+|\theta(n)|}$, in ascending order of frequency if n is odd, in descending order of frequency if n is even.

We set $F_0 = 15$.

6. Movement

The pieces described above are somewhat homogeneous. Musical entities have certain numerical quantities associated with them, for example, a musical note has a volume, a duration and a pitch. A sense of movement can be introduced by allowing these quantities to vary in a consistent fashion. We give an example of this, where the frequency, duration, and amplitudes of the harmonics of a note are exponentially dependent on a single parameter. The span and sets of notes of the phrases of this piece are also dependent on a single parameter.

Example 21. We are working in 31 tone equal temperament, as in Section 2. We begin by concatenating three relations corresponding to the linearly independent vectors $(-4, 4, -1, 0)$, $(1, 2, -3, 1)$ and $(-5, -1, -2, 4)$, in $\mathbb{Z}^{\{2, 3, 5, 7\}}$.

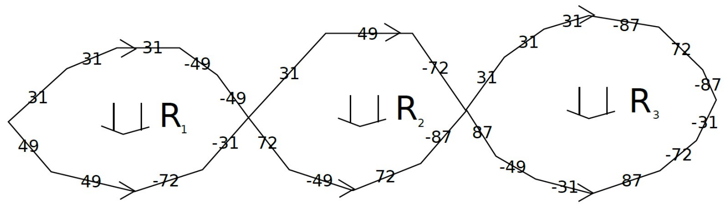

Figure 1. The three relations.

Taking the leftmost vertex as zero, and tracing our relations around in $\mathbb{Z}/31$, we obtain the following graph with labelled vertices:

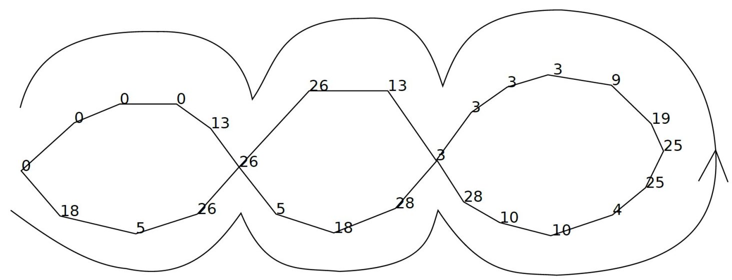

Figure 2. Elements of $\mathbb{Z}/31$ derived from the three relations.

Following the circumscribed path around, and ignoring repetitions, we obtain the following sequence:

$$a = (0, 18, 5, 26, 28, 3, 10, 4, 25, 19, 9, 13).$$

Let $\Sigma_i = \{a_1, a_2, a_3, \ldots, a_i\}$, for $i = 1, 2, 3, \ldots, 12$. Let

$$\Omega_i = \{n \in \mathbb{Z} \mid 0 \leq n \leq 27 + 4i,\ n \equiv x \bmod 31 \text{ for some } x \in \Sigma_i\}.$$

We play the set of notes Ω_i in ascending order, where the *j*th harmonic of $n \in \mathbb{Z}$ has frequency $440j.2^{\frac{n}{31}}$, amplitude $2^{\frac{n}{25j}}$ and duration $\frac{1}{8}.2^{\frac{n}{31}}$. Here, *j* runs from 1 to 7.

7. Rescaling Harmonics in 31 Tone Equal Temperament

Suppose we have $\zeta_2, \zeta_3, \zeta_5$ in the multiplicative subgroup $\langle 2, 3, 5, 7 \rangle$

of $\mathbb{R}^{\times}$, as in Section 4. Then we can approximate elements $\zeta_2, \zeta_3, \zeta_5$ by elements of $\langle 2^{\frac{1}{31}} \rangle$. Rescaling our Bach invention with these approximations in turn gives us a piece, played in 31 tone equal temperament.

Example 22. For a first example, we take $\zeta_2 = 2 = 2^{\frac{31}{31}}$, $\zeta_3 = 3 \approx 2^{\frac{49}{31}}$, $\zeta_5 = 5 \approx 2^{\frac{72}{31}}$. For a second example, we take $\zeta_2 = 3 \approx 2^{\frac{49}{31}}$, $\zeta_3 = 7 \approx 2^{\frac{87}{31}}$, $\zeta_5 = 21 \approx 2^{\frac{137}{31}}$. Note we do not approximate 21 by the closest possible power of $2^{\frac{1}{31}}$ available, namely $2^{\frac{49}{31}}.2^{\frac{87}{31}} = 2^{\frac{136}{31}}$. One effect of this is to introduce slight variations in pitch between notes that would be identical if we chose approximations $(2^{\frac{49}{31}}, 2^{\frac{87}{31}}, 2^{\frac{136}{31}})$ instead.

8. The Prime 11

In Sections 5 to 7, we have considered 31 tone equal temperament, which we were encouraged to contemplate by analysis of the second, third, fifth and seventh harmonics, in Section 2. One step beyond this is the eleventh harmonic, or the frequency ratio given by the prime 11.

It is possible to approximate the prime 11 in 31 tone equal temperament (we have $2^{\frac{107}{31}}/11 \sim 0.9946$). A better approximation can be found in 24 tone equal temperament, which also includes two copies of 12 tone equal temperament (we have $2^{\frac{83}{24}}/11 \sim 0.9992$). In 2 tone equal temperament, we have the weak approximation $2^{\frac{7}{2}}/11 \sim 1.0285$.

Here, we present examples involving the prime 11 in our frequency ratios.

Example 23. For our first example (rank 2), we take $\zeta_5 = \zeta_2^{-1}$ to be a relation. We take $\zeta_2 = 2$, $\zeta_3 = \frac{11}{10}$ and $\zeta_5 = \frac{1}{2}$.

For our second example (rank 3), we take $\zeta_2 = 2$, $\zeta_3 = 5$, $\zeta_5 = 11$. Thus, $(\zeta_2, \zeta_3, \zeta_5) \approx (2, 2^{\frac{7}{3}}, 2^{\frac{7}{2}})$, and our scale weakly approximates the whole tone scale.

We set our initial frequency to be $F = 264$ for the first example, and $F = 132$ for the second.

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